Stochastic Integration

(1) Constructions:

. Consider in $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. $(\mathcal{F}_t)$ is complete

① For mart. bdd in L^2:

Denote: H^2 is space of all conti. martingale M bdd in L^2 and $M_0 = 0$. Any two indistinguishable processes are identified. With inner product $(\cdot,\cdot)_{H^2}$ def by: $(M, N)_{H^2} = E(\langle M, N\rangle_\infty)$

Rmk: i) $M \in H^2 \Leftrightarrow M_0 = 0$, $E(\langle M, M\rangle_\infty) < \infty$.

and M is c.l.m.

Moreover, $\exists M_\infty \in L^2$. $M_t = E(M_\infty \mid \mathcal{F}_t)$

ii) Note: $(M, N)_{H^2} = E(\langle M, N\rangle_\infty) = E(M_\infty N_\infty)$

$(M, M)_{H^2} = 0 \Leftrightarrow M_\infty = 0 \Leftrightarrow M_t = 0.\ \forall t$

It's a true inner product.

Prop. H^2 equipped with $(\cdot,\cdot)_{H^2}$ is a Hilbert space.

Pf: Note: (M^n) is Cauchy seq in H^2.

$\Leftrightarrow (M^n_\infty)$ converges in L^2. to a limit Z.

By Doob's: $E(\sup_t (M^n_t - M^m_t)^2) \to 0$

$\Rightarrow (M^n_t)$ converges in L^2. Denote M_t.

1°) (M_t) has conti. sample paths.

$\exists (n_k)$. subseq. $E(\sum_1^\infty \sup_t |M_t^{n_k} - M_t^{n_{k+1}}|) \le$ (By Fubini's)

$$\sum_1^\infty E(\sup |M_t^{n_k} - M_t^{n_{k+1}}|^2)^{\frac{1}{2}} < \infty$$

$\Rightarrow \sum \sup_t |M_t^{n_k} - M_t^{n_{k+1}}| < \infty$. a.s. So: $M_t^{n_k} \xrightarrow{u} M_t$. a.s.

M is adapted. Since $(\mathcal{F}_t)$ is complete.

2°) $M_t^{n_k} = E(M_\infty^{n_k} | \mathcal{F}_t)$. Let $k \to \infty \Rightarrow M_t = E(Z|\mathcal{F}_t)$

$\Rightarrow M$ is conti. mart bdd in L^2. $\therefore M \in H^2$.

3°) $M_\infty = \lim M_\infty^{n_k} = Z$. a.s. by uniform converge of M.

So $M_\infty^n \longrightarrow M_\infty$ in L^2. i.e. $M^n \to M$ in H^2.

<u>Denote</u>: i) $\mathcal{P}$ is progressive σ-field on $\Omega \times \mathbb{R}_+$.

ii) For $M \in H^2$. $L^2(M) = L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP\, d\langle M,M\rangle)$

Identify H and H' two progressive process

if $H = H'$. $d\langle M,M\rangle_s$ - a.e. P-a.s.

<u>Rmk</u>: i) $dP\,d\langle M,M\rangle_s$ is a measure assign $A \in \mathcal{P}$

to $E(\int_A d\langle M,M\rangle_s)$. Its total mass

is $E(\langle M,M\rangle_\infty) = \|M\|_{H^2}$.

ii) $L^2(M)$ can be equipped with a inner

product: $(H,K)_{L^2(M)} = E(\int_0^\infty H_s K_s \, d\langle M,M\rangle_s)$

which is a Hilbert space.

<u>Def</u>: A elementary process is a progressive process of

form $H_s(\omega) = \sum_0^{p-1} H_{(i)}(\omega) I_{(t_i, t_{i+1}]}(s)$ where $0 = t_0 <$

$t_1 < \cdots < t_p$. $H_{(i)} \in \mathcal{F}_{t_i}$. bdd. $\forall\, 0 \le i \le p-1$.

Rmk: The set of all elementary processes form a linear subspace of $L^2(M)$. Denote by $\mathcal{E}$.

prop. $\forall M \in H^2$. $\mathcal{E}$ is dense in $L^2(M)$.

Pf: prove: if $K \in L^2(M)$. $K \perp \mathcal{E} \Rightarrow K = 0$.

Set $X_t = \int_0^t K_u \, d\langle M, M\rangle_u$ is FV process.

It makes sense: $E(\int_0^t |K_u| \, d\langle M, M\rangle_u) \leq \|K\|_{L^2(M)}^{\frac{1}{2}} \cdot \|M\|_{H^2}^{\frac{1}{2}}$

Besides. $\int_0^t |K_u| \, d\langle M, M\rangle_u < \infty$ a.s. $\forall t \geq 0 \Rightarrow X_t$ is well-def.

For $H_r(\omega) = F(\omega) I_{(s,t]}(r)$. $F \in \mathcal{F}_s$. if $(H, K)_{L^2(M)} = 0$

$\Rightarrow E(F \int_s^t K_u \, d\langle M, M\rangle_u) = 0$. i.e. $E(F(X_t - X_s)) = 0$

By X is adapted. $X_t \in L^1 \Rightarrow X$ is conti. mart.

Which implies $X = 0 \Rightarrow K = 0$. a.e. a.s.

Lemma. $M \in H^2 \Rightarrow M^T \in H^2$. for T stopping time.

Pf: $\langle M^T, M^T\rangle_\infty = \langle M, M\rangle_T \leq \langle M, M\rangle_\infty$.

Rmk: $H \in L^2(M) \Rightarrow I_{(0, T(\omega)]}(s) H_s(\omega)$.

Thm. (Construction for Sto-Integration)
$M \in H^2$. $\forall H = \sum_0^{p_n} H_{(i)} I_{(t_i, t_{i+1}]} \in \mathcal{E}$. $(H \cdot M)_t = \sum_0^{p_n} H_{(i)} \Delta M_{t_i \wedge t}^{t_{i+1} \wedge t}$.

defines a process $H \cdot M \in H^2$.

$H \mapsto H \cdot M$ can be extended to isometry from $L^2(M) \to H^2$.

Moreover, $H \cdot M$ is unique mart in H^2. satisfies: $\forall N \in H^2$.

$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle$.

For T stopping time. $(I_{(0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T$

Rmk: Denote $(H\cdot M)_t = \int_0^t H_s\, dM_s$ and call it the sto-Integration of H w.r.t. M.

Pf: 1°) Def of $H\cdot M$ for $H \in \mathcal{E}$ doesn't depend on the decomposition chosen of H.

2°) $H \mapsto H\cdot M$ is linear. Next, prove: it's isometry

Fix H. Set $M^i_t = H_{(i)}\, \Delta M^{t_{i+1}\wedge t}_{t_i \wedge t}$

check (M^i) is conti mart. $\Rightarrow M^i \in H^2$.

So: $H\cdot M = \sum_0^n M^i \in H^2$.

Note (M^i): orthogonal in $\langle\, ,\, \rangle$. bracket.

$\langle H\cdot M, H\cdot M\rangle_t = \sum \langle M^i, M^i\rangle$ (by def of $\langle\, ,\rangle$)

$= \sum H_{(i)}^2 (\langle M, M\rangle_{t_{i+1}\wedge t} - \langle M, M\rangle_{t_i\wedge t})$

$= \int_0^t H_s^2\, d\langle M, M\rangle_s$

$\Rightarrow \|H\cdot M\|^2_{H^2} = E(\int_0^\infty H_s^2\, d\langle M, M\rangle_s) = \|H\|^2_{L^2(M)}$

3°) Fix $N \in H^2$. if $H \in L^2(M)$. then by KW inequi.

$E(\int_0^\infty |H_s|\,|d\langle M, N\rangle_s|) \le \|H\|_{L^2(M)} \|N\|_{H^2} < \infty$

So $(H\cdot\langle M, N\rangle)_\infty$ is well-def and in L^1.

First. consider $H \in \mathcal{E}$. Then:

$\langle H\cdot M, N\rangle_t = \sum \langle M^i, N\rangle_t = \sum H_{(i)}\, \Delta \langle M, N\rangle^{t_{i+1}\wedge t}_{t_i\wedge t}$

$= \int_0^t H_s\, d\langle M, N\rangle_s = (H\cdot\langle M, N\rangle)_t$

By: $E(|\langle X, N\rangle_\infty|) \le \|N\|_{H^2} \|X\|_{H^2}$ KW inequi.

$\Rightarrow X \mapsto \langle X, N\rangle_\infty$ is BLO from H^2 to L^1.

For general $H \in L^2(M)$. consider $(H^n) \subset \mathcal{E} \xrightarrow{L^2(M)} H$.

$\Rightarrow$ In L^1 $\begin{cases} \langle H^n\cdot M, N\rangle_\infty \to \langle H\cdot M, N\rangle_\infty & \text{(by isometry)} \\ (H^n\cdot\langle M, N\rangle)_\infty \to (H\cdot\langle M, N\rangle)_\infty & \text{(by KW)} \end{cases}$

$\Rightarrow \langle H\cdot M, N\rangle_\infty = (H\cdot\langle M,N\rangle)_\infty$

Replace N by N^t. So: $\langle H\cdot M, N\rangle_t = (H\cdot\langle M,N\rangle)_t$.

4) To characterize $H\cdot M$. i.e. its uniqueness.

if $\langle H\cdot M - X, N\rangle = 0 \Rightarrow$ set $N = H\cdot M - X \in H^2$.

5) $\langle (H\cdot M)^T, N\rangle_t = \langle H\cdot M, N\rangle_{t\wedge T} = (H\cdot\langle M,N\rangle)_{T\wedge t}$

$= (I_{[0,T]} H\cdot\langle M,N\rangle)_t = \langle (I_{[0,T]}H)\cdot M, N\rangle_t$

Similarly for another statement.

Rmk: i) We can use $\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$. $\forall N \in H^2$

to define $H\cdot M$. Note: $N \longmapsto E((H\cdot\langle M,N\rangle)_\infty)$ is

BLF on H^2. (by KW inequi $\leq \|N\|_{H^2}\|H\|_{L^2(M)}$)

By Riesz Represent. $\exists$ unique $H\cdot M \in H^2$. st.

$(H\cdot M, N)_{H^2} = E((H\cdot\langle M,N\rangle)_\infty) = E(\langle H\cdot M, N\rangle_\infty)$

ii) By the notation of $H\cdot M$. We have:

$\langle \int_0^\cdot H_s\, dM_s, N\rangle_t = \int_0^t H_s\, d\langle M,N\rangle_s$ ---(*)

We interpret it by: "$\int$" commutes with "$\langle\cdot,\cdot\rangle$".

(it's Sto-integration, not common integration)

Cor. For $M \in H^2$. $H \in L^2(M)$. Then: $\langle H\cdot M, H\cdot M\rangle$

$= H\cdot(H\cdot\langle M,M\rangle) = H^2\cdot\langle M,M\rangle$.

generally. $\langle H\cdot M, K\cdot N\rangle = H\cdot(K\cdot\langle M,N\rangle)$

$= (HK)\cdot\langle M,N\rangle$. for $K \in L^2(N)$. $N \in H^2$.

Prop. $H \in L^2(M)$. If K is a progressive process. Then:

$KH \in L^2(M) \Leftrightarrow K \in L^2(H\cdot M)$. if these hold.

then. $(KH)\cdot M = K\cdot(H\cdot M)$.

Pf. $E(\int_0^\infty K_s^2 H_s^2 d\langle M,M\rangle_s) = E(\int_0^\infty K_s^2 d\langle H\cdot M, H\cdot M\rangle_s)$

For the latter, note that for $\forall N \in H^2$.

$\langle (KH)\cdot M, N\rangle = (KH)\cdot\langle M,N\rangle = \langle K\cdot(H\cdot M), N\rangle$

Moments of Sto-Integration:

Suppose $M, N \in H^2$. $H \in L^2(M)$. $K \in L^2(N)$. Then:

$E(\int_0^t H_r dM_r) = 0$. $E(\int_s^t H_r dM_r \mid \mathcal{F}_s) = 0$. by mart. prop.

$E((\int_0^t H_s dM_s)(\int_0^t K_s dN_s)) = E(\langle H\cdot M, K\cdot N\rangle_t)$

$= E(\int_0^t H_s K_s d\langle M,N\rangle_s)$

$\Rightarrow$ In particular, $E((H\cdot M)_t^2) = E(\int_0^t H_s^2 d\langle M,M\rangle_s)$

② For Local mart:

i) Now we extend Sto-integration to c.l.m's. M.

Denote: $L^2_{loc}(M) = \{H$ is progressive $\mid \int_0^t H_s^2 d\langle M,M\rangle_s < \infty$. $\forall t$. a.s$\}$

$L^2(M) = \{H$ is progressive $\mid \int_0^\infty H_s^2 d\langle M,M\rangle_s < \infty\}$.

Thm. For M is c.l.m. $\forall H \in L^2_{loc}(M)$. $\exists$ unique c.l.m with initial value 0. denote by $H\cdot M$. s.t. $\forall N$ c.l.m.

$\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$. Besides.

i) If T is stopping time. Then: $(I_{[0,T]}H)\cdot M = (H\cdot M)^T$

$= H\cdot M^T$.

ii) K is progressive. Then: $K \in L^2_{loc}(H\cdot M) \Leftrightarrow KH \in L^2_{loc}(M)$

And then: $H\cdot(K\cdot M) = (HK)\cdot M$. if these hold.

iii) If $M \in \mathcal{H}^2$. $H \in L^2(M)$. Then the def is consistent with def of $H \cdot M$ before.

<u>Pf</u>: 1°) Assume: $M_0 = 0$. Or we can write $M = M_0 + M'$

set $H \cdot M = H \cdot M'$.

$\int_0^t H_s^2 \, d\langle M, M\rangle_s < \infty$. $\forall t \geq 0$. $\forall w \in \Omega$. by modify H. set $H = 0$ on $N = \{\square = \infty\}$.

Set: $T_n = \inf\{t \geq 0 \mid \int_0^t (1 + H_s^2) \, d\langle M, M\rangle_s \geq n\}$.

$\Rightarrow M^{T_n} \in \mathcal{H}^2$. and $\langle H \cdot M^{T_n}, H \cdot M^{T_n}\rangle_\infty \leq n$. so: $H \in L^2(M^{T_n})$

2°) Note: for $m > n$. $H \cdot M^{T_n} = (H \cdot M^{T_m})^{T_n}$.

$\Rightarrow \exists$ unique process denoted by $H \cdot M$. st.

$(H \cdot M)^{T_n} = H \cdot M^{T_n}$. $\forall n \in \mathbb{N}$

$H \cdot M = \lim_n H \cdot M^{T_n}$ conti and adapted.

Since $(H \cdot M)^{T_n} \in \mathcal{H}^2 \Rightarrow H \cdot M$ is c.l.m.

3°) Fix N is c.l.m $N_0 = 0$. Set $\tilde{T}_n = \inf\{t \geq 0 \mid |N_t| \geq n\}$

$S_n = T_n \wedge \tilde{T}_n$. Check: $\langle H \cdot M, N\rangle^{S_n} = (H \cdot \langle M, N\rangle)^{S_n}$

Let $n \to \infty$. For uniqueness. argue as before.

4°) i). ii) is identical as before. follows from the characterization of $H \cdot M$.

For iii): Note: $\langle H \cdot M, H \cdot M\rangle = H^2 \cdot \langle M, M\rangle$

$\Rightarrow H \cdot M \in \mathcal{H}^2$. And the charac. show consistency

<u>Rmk</u>: We denote $(H \cdot M)_t = \int_0^t H_s \, dM_s$ as before.

The formulars (*) stay valid in c.l.m's.

ii) Connection with

Wiener Integral:

For B is $(\mathcal{F}_t)$-SBM. $h \in L^2(\mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+}, dt)$.

Def: Wiener integral $\int_0^t h(s)\, dB_s = G(h I_{(0,t]})$.

prop. It coincides with sto-integral $(h \cdot B)_t$.

Pf: It holds for $h = I_{(a,b]}$.

Note G is isometry. Approxi h by simple func's.

iii) Moments of integrals:

If M is c.l.m. $H \in L^2_{loc}(M)$. for $t \in \mathbb{R}^+$. under condition $E(\int_0^t H_s^2\, d\langle M, M\rangle_s) < \infty$. Then we obtain: $(H \cdot M)^t \in H^2$. As before. it satisfies:

$$\begin{cases} E(\int_0^t H_s\, dM_s) = 0 \\ E((\int_0^t H_s\, dM_s)^2) = E(\int_0^t H_s^2\, d\langle M, M\rangle_s) \end{cases}$$

and mart property for $0 \le s \le t$:

$E(\int_s^t H_s\, dM_s \mid \mathcal{F}_s) = 0$.

Rmk: If the condition doesn't hold. We still have:

$E((\int_0^t H_s\, dM_s)^2) \le E(\int_0^t H_s^2\, d\langle M, M\rangle_s)$

Since "if RHS $= \infty$" is trivial.

③ For Semimartingales:

Finally. we extend the sto-integral to conti. semimart.

Def: i) A progressive process H is locally bdd if:

$\forall t \geq 0$. $\sup_{s \leq t} |H_s| < \infty$. a.s.

Rmk: i) $\forall$ conti. adapted process is locally bdd.

ii) If H is locally bdd. Then $\forall$ V FV process, we have: $\forall t \geq 0$. $\int_0^t |H_s||dV_s| < \infty$. a.s. and $H \in L^2_{loc}(m)$ for m is c.l.m.

ii) For $X = M + V$ conti. semimart. H is locally bdd. Sto-integral $H \cdot X$ is conti. semimart. define by $H \cdot X = H \cdot M + H \cdot V$. Denote $H \cdot X = \int_0^\cdot H_s dX_s$.

prop. i) $(H, X) \mapsto H \cdot X$ is bilinear.

ii) $H \cdot (K \cdot X) = (HK) \cdot X$ if H, K are locally bdd.

iii) For every stopping time. $(H \cdot X)^T = H \cdot X^T = (I_{[0,T]} H) \cdot X$.

iv) If X is c.l.m or FV process. Then: so does $H \cdot X$

v) If H is form: $H = \sum_0^{p-1} H_{(i)} I_{(t_i, t_{i+1}]}(s)$. $H_{(i)} \in \mathcal{F}_{t_i}$. Then:

$(H \cdot X)_t = \sum_0^{p-1} H_{(i)} (X_{t_{i+1} \wedge t} - X_{t_i \wedge t})$.

Pf: i) - iv) follows from sto-integral of FV. c.l.m.

v) Enough to prove c.l.m case (FV have def)

Assume: $M_0 = 0$. stop it at suitable time. $\Rightarrow M \in H^2$.

Set $T_n = \inf\{t \geq 0 \mid |H_t| \geq n\} = \inf\{t_i \mid |H_{(i)}| \geq n\}$.

which is for circumvent that $H_{(i)}$ isn't bdd.

$\Rightarrow H I_{[0,T_n]} = \sum H^n_{(i)} I_{(t_i, t_{i+1}]}(s)$. $H^n_{(i)} = H_{(i)} I_{[T_n > t_i]}$

is elementary process. Since $|H^n_{(i)}| \le n$.

Besides. $(H \cdot M)_{t \wedge T_n} = (H I_{[0,T_n]} \cdot M)_t = \sum H^n_{(i)} \Delta M^{i+1}_i$.

Since $T_n \uparrow \infty$. Set $n \to \infty$.

④ Convergence:

Thm. For $X = M + V$ conti. semimart. $(H^n)_{n \ge 1}$ is seq of locally bdd progressive process. K is nonnegative progressive process. If following holds a.s.:

i) $H^n_s \xrightarrow{n \to \infty} H_s$. $\forall s \in [0,t]$. fix t.

ii) $|H^n_s| \le K_s$. $\forall n$. $\forall s \in [0,t]$.

iii) $\int_0^t K_s^2 \, d\langle M, M\rangle_s < \infty$. $\int_0^t K_s |dV_s| < \infty$.

Then: $\int_0^t H^n_s \, dX_s \xrightarrow[p]{n \to \infty} \int_0^t H_s \, dX_s$.

Rmk: i) if K is locally bdd. then iii) holds automatically.

ii) Conditions i). ii) can be assumed: $s \in [0,t]$. $d\langle M, M\rangle_s$ - a.e and $|dV_s|$ - a.e.

Pf: 1°) $\int_0^t H^n_s \, dV_s \xrightarrow{n \to \infty} \int_0^t H_s \, dV_s$ P-a.s. by DCT.

2°) For cl.m part:

Set $T_p = \inf\{ s \in [0,t] \mid \int_0^r K_s^2 \, d\langle M, M\rangle_s \ge p \} \wedge t \uparrow t$.

$\Rightarrow (H^n - H) \cdot M^{T_p} \in H^2$. by ii). So:

$E((\int_0^{T_p} (H^n_s - H_s) \, dM_s)^2) = E(\int_0^{T_p} (H^n_s - H_s)^2 \, d\langle M, M\rangle_s)$

Let $n \to \infty$ by DCT. With $P(T_p = t) \xrightarrow{p \to \infty} 1$

Prop. X is conti. semimart. H is adapted conti. process.

Then: $\forall t>0$. $\forall\ 0=t_0^n<t_1^n<\cdots<t_{p_n}^n=t$ subdivision of $[0,t]$. whose mesh $\to 0$. we have:

$$\lim_n \sum_0^{p_n-1} H_{t_i^n}(X_{t_{i+1}^n}-X_{t_i^n}) = \int_0^t H_s\,dX_s \text{ in prob.}$$

Pf: Set $H_s^n = H_0 I_{\{s=0\}} + \sum_0^{p_n-1} H_{t_i^n} I_{\{t_i^n<s\le t_{i+1}^n\}}$. progressive.

Take $K_s = \max_{[0,s]}|H_r|$. dominated H_s. locally bdd

Then by DCT. Thm above $\Rightarrow \int_0^t H_s^n\,dX_s \to \int_0^t H_s\,dX_s$.

Rmk: The prop fails if $LHS = \sum H_{t_{i+1}^n}(X_{t_{i+1}^n}-X_{t_i^n})$

i.e. evaluate H at the right end rather than left end of $(t_i^n, t_{i+1}^n]$.

(2) Itô's Formular:

① Thm: (Itô's Formula)

$X^1, \cdots X^p$ are conti. semimart. $F\in C^2(\mathbb{R}^p, \mathbb{R})$

Then $\forall t\ge 0$. P-a.s. $F(X_t^1\cdots X_t^p) = F(X_0^1\cdots X_0^p)$

$$\sum_1^p \int_0^t \frac{\partial F}{\partial x^i}(X_s^1\cdots X_s^p)\,dX_s^i + \frac{1}{2}\sum_{i,j}^p \int_0^t \frac{\partial^2 F}{\partial x^i\partial x^j}(X_s^1\cdots X_s^p)\,d\langle X^i, X^j\rangle_s$$

Pf: 1°) $p=1$. $X=X^1$.

Consider $0=t_0^n<t_1^n\cdots<t_{p_n}^n=t$ subdivision of $[0,t]$ whose mesh $\to 0$.

From: $F(X_t) = F(X_0) + \sum F(X_{t_{i+1}^n}) - F(X_{t_i^n})$

By Talyor expansion on: $\theta\in[0,1]\mapsto F(X_{t_i^n}+\theta\Delta X_i^{i+1})$

$\Rightarrow F(X_{t_{i+1}^n}) - F(X_{t_i^n}) = F'(X_{t_i^n})(X_{t_{i+1}^n}-X_{t_i^n}) + \frac{1}{2} f_{n,i}\,\Delta^2 X_i^{i+1}$

By prop above. The first term $\xrightarrow{P} \int_0^t F'(X_s)\,dX_s$.

select a subseq guarantee it holds a.s.

For the second term:

prove: $\sum f_{n,i} \Delta^2 X_i^{i+1} \xrightarrow{p} \int_0^t F''(X_s)\, d\langle X, X\rangle_s$

where $f_{n,i} = F''(X_{t_i^n} + c(X_{t_{i+1}^n} - X_{t_i^n}))$. $c \in [0,1]$.

$\sup_i |f_{n,i} - F''(X_{t_i^n})| \leq \sup_i \left(\sup_{[X_{t_i^n}, X_{t_{i+1}^n}]} |F''(x) - F''(X_{t_i^n})| \right)$

$\rightarrow 0$. a.s. by unif-conti of F''.

Combined with $\sum (X_{t_{i+1}^n} - X_{t_i^n})^2 \xrightarrow{p} \langle X, X\rangle_t$, $n \to \infty$

$\Rightarrow |\sum f_{n,i} \Delta^2 X_i^{i+1} - \sum F''(X_{t_i^n}) \Delta^2 X_i^{i+1}| \xrightarrow{p} 0$, $n \to \infty$.

prove: $\sum F''(X_{t_i^n}) \Delta^2 X_i^{i+1} \xrightarrow{p} \int_0^t F''(X_s)\, d\langle X, X\rangle_s$.

LHS $= \int_{[0,t]} F''(X_s)\, \mu_n(ds)$. $\mu_n = \sum (X_{t_{i+1}^n} - X_{t_i^n})^2 \delta_{t_i^n}$

consider D countable dense in $[0,t]$. $(t_i^n) \subset D$. $\forall n$.

$\forall r \in D$. $\mu_n[0,r] \xrightarrow[p]{n\to\infty} \langle X, X\rangle_r$.

By diagonal argument. $\exists (n_k)$. $\mu_{n_k}[0,r] \xrightarrow[a.s.]{k\to\infty} \langle X, X\rangle_r$. $\forall r \in D$

$\Rightarrow$ LHS $\xrightarrow{a.s.} \int_0^t F''(X_s)\, d\langle X, X\rangle_s$.

2') For general case:

Apply Taylor on: $\theta \in [0,1] \mapsto F(X_{t_i^n}^1 + \theta \Delta X^{1\, i+1}_{i}, \cdots X_{t_i^n}^p + \theta \Delta \cdots)$

<u>Rmk</u>: i) Most convergences we meet is converge in pr.

Sometimes, it doesn't matter. like in Itô's Formula "n" doesn't involve in. So we can select subseq to guarantee a.s.-convergence.

Sometimes. the seq is u.i. So we can imply L^1-convergence.

ii) A special case of Itô's Formula is integration by part: consider $p=2$ $F(x,y)=xy$.

$X_tY_t = X_0Y_0 + \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \langle X,Y\rangle_t$

iii) Set $X=Y$ above. it shows: $X_t^2 - \langle X,X\rangle_t$

$= X_0^2 + 2\int_0^t X_s\,dX_s$. c.l.m. direct form.

iv) Note that we prove Itô's Formula locally.

if $F \in C^2(U)$, $U \underset{open}{\subseteq} \mathbb{R}^p$. then $\forall V \subset\subset U$

open set. Set $T_V = \inf\{t \geq 0 \mid (X_t^1, \cdots X_t^p) \notin V\}$

and $\exists\, G = F$ in $\bar{V}$. $G \in C^2(\mathbb{R}^p)$.

$\Rightarrow$ Apply Itô's Formular on $G(X^1_{t\wedge T_V}, \cdots X^p_{t\wedge T_V})$

If in addition, we know $(X_t^1, \cdots X_t^p) \in U$. a.s.

Let $V \uparrow U$. $\Rightarrow$ We obtain Itô's Formula for $F(\vec{X}_t)$

stays valid. e.g. $F(x) = \log x$.

Def: A random process takes values in $\mathbb{C}$ is complex c.l.m

if its imaginary and real part are c.l.m's

prop. M is c.l.m. for $\lambda \in \mathbb{C}$. Let $\mathcal{E}(\lambda M)_t = e^{\lambda M_t - \frac{\lambda^2}{2}\langle M,M\rangle_t}$

Then $\mathcal{E}(\lambda M)$ is complex c.l.m. having the form:

$\mathcal{E}(\lambda M)_t = e^{\lambda M_0} + \lambda \int_0^t \mathcal{E}(\lambda M)_s\,dM_s$.

Pf: $F(r,x) = e^{\lambda x - \frac{\lambda^2}{2} r} \in C^2$ satisfies $\begin{cases} \frac{\partial F}{\partial x} = \lambda F \\ \frac{\partial F}{\partial r} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0 \end{cases}$

Apply Itô on $(M_t, [M]_t)$.

Rmk: We can construct ch.f by $\mathcal{E}(\lambda M)$!

(3) Applications of Itô's Formular:

① Lévy's Charac. of BMs:

Thm. For $X = (X^1, \cdots X^d)$ adapted. conti. Follows equi.:

i) X is d-dim $(\mathcal{F}_t)$-BM.

ii) $X^1, \cdots X^d$ are c.l.m's and $\langle X^i, X^j \rangle_t = \delta_{ij} t$. $\forall i, j$.

Rmk: In particular, c.l.m. M is $(\mathcal{F}_t)$-BM.

$\iff \langle M, M \rangle_t = t$, $\forall t \geq 0 \iff M_t^2 - t$ is c.l.m.

Pf: i) $\Rightarrow$ ii) We have proved. For ii) $\Rightarrow$ i).

For $\xi \in \mathbb{R}^d$. $\xi \cdot X_t = \sum \xi_i X_t^i$. is c.l.m.

With QV: $\langle \xi \cdot X, \xi \cdot X \rangle_t = \sum \xi_i \xi_j \langle X^i, X^j \rangle_t = |\xi|^2 t$

$\Rightarrow \mathcal{E}(i\xi X) = e^{i\xi \cdot X + \frac{1}{2}|\xi|^2 t}$ is complex. c.l.m. bdd

on every cpt interval. So it's true mart.

$\Rightarrow E(e^{i\xi \cdot (X_t - X_s)} | \mathcal{F}_s) = e^{-\frac{1}{2}|\xi|^2 (t-s)}$ by mart prop.

So. $\forall A \in \mathcal{F}_s$. $E(I_A e^{i\xi \cdot (X_t - X_s)}) = P(A) e^{-\frac{1}{2}|\xi|^2 (t-s)}$

If $A = \Omega$. Then we have: $X_t - X_s \sim N(0, (t-s)I)$

Furthermore. if $P(A) > 0$, then $E(e^{i\xi \cdot (X_t - X_s)} | A)$

$= e^{-\frac{1}{2}|\xi|^2 (t-s)}$, i.e. $X_t - X_s | A \sim X_t - X_s$. $\forall A \in \mathcal{F}_s$

$\Rightarrow X_t - X_s$ is indpt with $\mathcal{F}_s$.

② Conti. Mart. as Time-changed BM:

Thm. (Dambis - Dubins - Schwarz)

If M is c.l.m. st. $\langle M, M \rangle_\infty = \infty$ a.s. Then $\exists$ BM

$(\beta_s)_{s \geq 0}$. st. a.s. $\forall t \geq 0$. $M_t = \beta_{\langle M, M \rangle_t}$.

Lemma: For M is c.l.m. We have a.s. for $0 \le a < b$:

$M_t = M_a.\ \forall t \in [a, b] \iff \langle M, M \rangle_b = \langle M, M \rangle_a$

Pf: By conti of M. prove for $0 \le a < b$. $a, b \in \mathbb{Q}$.

So we can fix. a, b. prove: a.s.

$\{M_t = M_a.\ \forall t \in [a, b]\} = \{\langle M, M \rangle_a = \langle M, M \rangle_b\}$.

"$\subset$" is trivial. for "$\supset$":

Set $N_t = M_t - M_{t \wedge a}$ $\quad T_\varepsilon = \inf\{t \ge 0 \mid \langle N, N \rangle_t \ge \varepsilon\}$.

$\Rightarrow \langle N, N \rangle_t = \langle M, M \rangle_t - \langle M, M \rangle_{t \wedge a}$

$N^{T_\varepsilon} \in H^2$. So: $E(N_{t \wedge T_\varepsilon}^2) = E(\langle N, N \rangle_{t \wedge T_\varepsilon}) \le \varepsilon$

Note $A = \{\langle M, M \rangle_a = \langle M, M \rangle_b\} \subset \{T_\varepsilon > b\}$.

So for $t \le b$. $E(N_t^2 I_A) = E(N_{t \wedge T_\varepsilon}^2 I_A) \le \varepsilon \to 0$

$\Rightarrow N_t = 0$ a.s. for $a \le t \le b$. on A.

Rmk: It's intuitive that if TV of Func $= 0$ then it's const. Consider $M_t - M_{t \wedge a}$ is like set $\tilde{f} = f(x) - f(0)$. but we also need to guarantee N_t is c.l.m.

Pf of Thm: 1') First we assume $M_0 = 0$.

Set $\tau_r = \inf\{t \ge 0 \mid \langle M, M \rangle_t \ge r\}$. $\tau_r < \infty$. $\forall r$.

by redefine $\tau_r(\omega) = 0$ for $\omega \in \{\langle M, M \rangle_\infty < \infty\}$

since $(\mathcal{F}_t)$ is complete. τ_r is stopping time

2') $r \mapsto \tau_r(\omega)$ is left-conti $\uparrow$. (nondecreasing)

So it's has right-limit at $\forall r \ge 0$.

Denote by $\tau_{r+} = \inf\{t \ge 0 \mid \langle M, M \rangle_t > r\}$

$\tau_{r+} = 0$ on $\{\langle M, M \rangle_\infty < \infty\}$.

3°) Set $\beta_r = M_{\tau_r}$ adapted to $\mathcal{F}_{\tau_r} \triangleq \mathcal{G}_r$. complete.

By 1°). 2°). conti of M. $r \mapsto \beta_r(w)$ is left-conti

and right-limit : $\beta_{r+} = \lim_{s \downarrow r} \beta_s = M_{\tau_{r+}}$.

By Lemma : $\beta_{r+} = \beta_r$. Since $\langle M, M\rangle_{\tau_r} = \langle M, M\rangle_{\tau_{r+}} = r$

$\Rightarrow$ path of β is conti (redefine it on null set)

4°) Prove : β_s. $\beta_s^2 - s$ are mart. w.r.t $(\mathcal{G}_s)$.

Note : M^{τ_n}. $(M^{\tau_n})^2 - \langle M, M\rangle^{\tau_n}$ are u.i. marts.

implies $0 \le r \le s < n$. $E(\beta_s | \mathcal{G}_r) = \beta_r$ ... used by M^{τ_n}...

By Lévy Charac. $\Rightarrow$ β is BM w.r.t $\mathcal{G}_r$.

5°) Prove : $M_{\tau_{\langle M,M\rangle_t}} = M_t$.

Since $\tau_{\langle M,M\rangle_t} \le t \le \tau_{\langle M,M\rangle_t +}$. combined with :

$\langle M, M\rangle_{\tau_\square} = \langle M, M\rangle_{\tau_{\square +}}$ $\Rightarrow$ $M_t = M_{\tau_\square}$. $\forall t \in [\tau_\square, \tau_{\square+}]$

6°) General. for $M_0 \ne 0$. set $M = M_0 + M'$.

Apply the argument on M'. $M'_t = \beta'_{\langle M', M'\rangle_t}$

$\Rightarrow$ $\beta_s = M_0 + \beta'_s$ is also BM (β' indept with $\mathcal{F}_0$)

To remove "$\langle M, M\rangle_\infty = \infty$". consider in a larger space :

<u>Def</u>: Enlargement of filtrated prob. space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$

is $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{P})$ with $\pi : \tilde{\Omega} \to \Omega$. s.t.

$\pi^{-1}(\mathcal{F}_t) \subset \tilde{\mathcal{F}}_t$. $\forall t$. $\tilde{P} \circ \pi^{-1} = P$

<u>Rmk</u>: Process X on Ω may be viewed as def on $\tilde{\Omega}$

by set : $X(\tilde{w}) = X(w)$ for $\pi(\tilde{w}) = w$.

Thm. Exists enlargement $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_{\tau_t}), P)$ and BM $\tilde{\beta}$ on $\tilde{\Omega}$, indept with M, c.l.m. st.

$$\beta_t = \begin{cases} M_{\tau_t} & \text{if } t < \langle M, M\rangle_\infty \\ M_\infty + \tilde{\beta}_{t - \langle M, M\rangle_\infty} & \text{if } t \geq \langle M, M\rangle_\infty \end{cases} \quad \text{is SBM.}$$

Besides, $W_t =: M_{\tau_t} I_{\{t < \langle M, M\rangle_\infty\}} + M_\infty I_{\{t \geq \langle M, M\rangle_\infty\}}$ is a $(\tilde{\mathcal{F}}_t)$ - BM stopped at $\langle M, M\rangle_\infty$.

prop. For M, N, c.l.m's. st. $M_0 = N_0 = 0$, $M_t = \beta_{\langle M, M\rangle_t}$, $N_t = \gamma_{\langle N, N\rangle_t}$ where β, γ are BMs correspond M, N respectively.

If i) $\langle M, M\rangle_t = \langle N, N\rangle_t$, $\forall t \geq 0$, a.s.

ii) $\langle M, M\rangle_\infty = \langle N, N\rangle_\infty = \infty$, a.s. iii) $\langle M, N\rangle_t = 0$, $\forall t$, a.s.

Then: β is indept with γ.

Pf: Note: $\begin{cases} \beta_r = M_{\tau_r} \\ \gamma_r = N_{\tau_r} \end{cases}$ $\tau_r = \inf\{t \geq 0 \mid \langle M, M\rangle_t \geq r\} = \inf\{t \geq 0 \mid \langle N, N\rangle_t \geq r\}$.

By iii) $M \perp N \Rightarrow M_t N_t$ is c.l.m. Then:

$M_t^{\tau_n} N_t^{\tau_n}$ is u.i. mart. Apply Optional Stop Thm.

$\Rightarrow E(\beta_s \gamma_s \mid \mathcal{G}_r) = E(M_{\tau_s}^{\tau_n} N_{\tau_s}^{\tau_n} \mid \mathcal{F}_{\tau_r}) = \beta_r \gamma_r$.

So $\beta_r \gamma_r$ is $(\mathcal{G}_r)$ - mart. $\Rightarrow \langle \beta, \gamma\rangle = 0$, a.s.

By Lévy Charac.: (β, γ) is 2-dim BM.

③ BDG Inequality:

Remark: For M, c.l.m, $M_t^* =: \sup_{s \leq t} |M_s|$

<u>Thm.</u> (Burkholder - Davis - Gundy)

$\forall p>0$. $\exists C_p, c_p>0$. s.t. for $\forall$ c.l.m. M.

with $M_0=0$ and every stopping time T.

$$c_p E(\langle M,M\rangle_T^{\frac{p}{2}}) \le E((M_T^*)^p) \le C_p E(\langle M,M\rangle_T^{\frac{p}{2}})$$

<u>Pf:</u> 1°) Replace M by M^T. only need to

prove it when $T=\infty$

Assume M is bdd by replace M by

M^{T_n}. $T_n=\inf\{t\ge0 \mid |M_t|\ge n\} \Rightarrow M\in H^2$.

Then set $n\to\infty$. by MCT.

2°) Right side. $p\ge 2$.

Apply Itô's Formula on $|x|^p \in C^2(\mathbb{R}^1)$

Then take expectation. By Hölder and Doob's

3°) Left side. $p\ge 4$.

From $\langle M,M\rangle_t = M_t^2 - 2\int_0^t M_s\,dM_s$. by C_p's inequi.

$$\Rightarrow E(\langle M,M\rangle_\infty^{\frac{p}{2}}) \le a_p\left(E(M_\infty^{*p}) + E(|\int_0^\infty M_s\,dM_s|^{\frac{p}{2}})\right)$$

$$\le a_p\left(E(M_\infty^{*p}) + E((\int_0^\infty M_s^2\,d\langle M,M\rangle_s)^{\frac{p}{4}})\right)$$

$$\le a_p\left(E(M_\infty^{*p}) + (E(M_\infty^{*p})E(\langle M,M\rangle_\infty^{\frac{p}{2}}))^{\frac{1}{2}}\right)$$

set $x=E(\langle M,M\rangle_\infty^{\frac{p}{2}})^{\frac{1}{2}}$, $y=E(M_\infty^{*p})^{\frac{1}{2}}$.

solve $x^2-a_pxy-a_py^2\le 0 \Rightarrow c_px\le y$.

For other parts, we introduce two Lemmas:

Def: (Dominated Relation)

A positive. adapted. right-conti process X is dominated by an increasing process A if:

$E(X_T | \mathcal{F}_0) \le E(A_T | \mathcal{F}_0)$. for $\forall$ bdd stopping time. T

Lemma. If X is dominated by A. conti. Then for $x, y > 0$

we have: $P(X_\infty^* > x, A_\infty \le y) \le \frac{1}{x} E(A_\infty \wedge y)$.

Pf: It suffices to prove it when $P(A_0 \le y) = 1$.

by replace P by $P' = P(\cdot | A_0 \le y)$. Condition still holds.

Assume X_∞ exists by stopping X at proper time.

Then apply Fatou's Lemma to obtain conclusion.

Let: $R = \inf\{t \ge 0 | A_t > y\}$. $S = \inf\{t \ge 0 | X_t > x\}$

$\Rightarrow \{A_\infty \le y\} = \{R = \infty\}$. Then we have:

$$
\begin{aligned}
P(X_\infty^* > x, A_\infty \le y) &= P(X_\infty^* > x, R = \infty) \\
&\le P(X_S \ge x, S < \infty, R = \infty) \\
&\le P(X_{S \wedge R} \ge x) \le \frac{1}{x} E(X_{S \wedge R}) \\
&\le \frac{1}{x} E(A_{S \wedge R}) \le \frac{1}{x} E(A_\infty \wedge y)
\end{aligned}
$$

Lemma. If X is dominated by A conti. Then: $\forall k \in (0,1)$

We have: $E(X_\infty^{*k}) \le \frac{2-k}{1-k} E(A_\infty^k)$

Pf: Set $F: \mathbb{R}^+ \to \mathbb{R}^+$. conti. $\uparrow$. $F(0) = 0$.

$$
\begin{aligned}
E(F(X_\infty^*)) &= E\left(\int_0^\infty I_{\{X_\infty^* > x\}} dF(x)\right) \quad \text{. by Fubini.} \\
&\le \int_0^\infty \left(P(X_\infty^* > x, A_\infty \le x) + P(A_\infty > x)\right) dF
\end{aligned}
$$

$$\overset{\text{(lemma)}}{\leq} \int_0^\infty \left(\frac{1}{x} E(A_\infty \wedge x) + P(A_\infty > x) \right) dF(x)$$

$$= \int_0^\infty \left(\frac{1}{x} E(A_\infty I_{\{A_\infty \leq x\}}) + 2P(A_\infty > x) \right) dF(x)$$

$$\overset{\text{(Fubini)}}{=} 2E(F(A_\infty)) + E\left(A_\infty \int_{A_\infty}^\infty \frac{dF(x)}{x}\right)$$

Then set $F = x^k$. we obtain the result.

Return to the pf:

4°) Right side. $0 < p < 2$.

Set $X = M^2$. $A = \langle M, M \rangle$. $\Rightarrow E(M_T^2 | \mathcal{F}_0) = E(\langle M, M \rangle_T | \mathcal{F}_0)$

$\Rightarrow E(M_\infty^{*\,2k}) \leq \frac{2-k}{1-k} E(\langle M, M \rangle_\infty^k)$. $k \in (0,1)$

5°) Left side, $0 < p < 4$

Set $X = \langle M, M \rangle^2$. $A = C_4 M^{*4}$. Consider $M I_{A_0}$, $A_0 \in \mathcal{F}_0$.

$\Rightarrow E(\langle M, M \rangle_T^2 | \mathcal{F}_0) \leq C_4 E(M_T^{*4} | \mathcal{F}_0)$ by 3°).

So: $E(\langle M, M \rangle_\infty^{2k}) \leq \frac{2-k}{1-k} C_4^k E(M_\infty^{*4k})$. $k \in (0,1)$

Cr. M is c.l.m. with $M_0 = 0$. Then:

$E(\langle M, M \rangle_\infty^{\frac{1}{2}}) < \infty \Rightarrow M$ is u.i. mart.

Pf: $M_\infty^* \in L^1$. dominates M.

Rmk: i) It's weaker than $E(\langle M, M \rangle_\infty) < \infty$

which implies: $M \in H^2$.

ii) Apply on Ito-integral: $\int_0^t H_s dM_s$.

If $E\left(\left(\int_0^t H_s^2 d\langle M, M \rangle_s\right)^{\frac{1}{2}}\right) < \infty$. $\forall t \geq 0$.

Then: $\int_0^t H_s dM_s$ is a mart.

($\forall$ $(H \cdot M)^t$ is u.i. mart. Check mart prop)

(4) Representation of Marts :

Suppose the filtration $(\mathcal{F}_t)$ on Ω is complete canonical filtration of a SBm $(B_t)_{t\geq 0}$

① Represent Thm :

Lemma : $V = \text{span}\{ e^{i\sum_1^n \lambda_i (B_{t_i} - B_{t_{i-1}})} \mid 0 = t_0 < t_1 < \cdots < t_n,$
$\lambda_i \in \mathbb{R}^1,\ 1 \leq i \leq n \}$ is dense in $L^2_{\mathbb{C}}(\Omega, \mathcal{F}_\infty, P)$

Pf. Prove: $\forall Z \in L^2_{\mathbb{C}}(\Omega, \mathcal{F}_\infty, P)$, s.t. $E(Z e^{i\sum_1^n \lambda_i (B_{t_i} - B_{t_{i-1}})}) = 0$
for $\forall \lambda_i \in \mathbb{R}^1, (t_i)$. $\Rightarrow Z = 0$ a.s.

By Weierstrass Approxi. $(e^{i\sum_1^n \lambda_i x_i}) \xrightarrow{u} \forall f \in C(\mathbb{R}^n)$

$\overset{DCT}{\Rightarrow} E(Z f(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})) = 0$. $\forall f \in C_0(\mathbb{R}^n)$

$C_0 \xrightarrow[u]{\text{Approxi}}$ simple func. I_A, $A \in \sigma(B_{t_i}, 1 \leq i \leq n)$

$\overset{DCT}{\Rightarrow} E(Z I_A) = 0$. $\forall A \in \sigma(B_t, t \geq 0) = \mathcal{F}_\infty$ $\therefore Z = 0$
follows from Monotone Class argument.

Thm. $\forall Z \in L^2(\Omega, \mathcal{F}_\infty, P)$. $\exists$ unique progressive process $h \in L^2(B)$. s.t. $Z = E(Z) + \int_0^\infty h_s \, dB_s$

Pf. i) Uniqueness of h:

$E(\int_0^\infty (h_s - h_s')^2 ds) = E((\int_0^\infty h_s \, dB_s - \int_0^\infty h_s' \, dB_s)^2) = 0$

2) Set $\mathcal{H}$ is Ls of all $Z \in L^2(\Omega, \mathcal{F}_\infty, P)$ which satisfies the statement

Note if $Z \in \mathcal{H}$, then:

$E(Z^2) = E(Z)^2 + E(\int_0^\infty h_s^2 \, ds)$

$\Rightarrow \mathcal{H}$ is closed subspace of $L^2(\Omega, \mathcal{F}_\infty, P)$

3°) Prove $\mathcal{H}$ is dense in $L^2(\Omega, \mathcal{F}_\infty, P)$

Consider $(\lambda_i)_1^n \subset \mathbb{R}^1$. $0 = t_0 < t_1 < \cdots < t_n$.

Set $f(s) = \sum_1^n \lambda_i I_{(t_{i-1}, t_i]}$. $\mathcal{E}_t^f = \mathcal{E}(i\int_0^\cdot f(s)\,dB_s)$

By prop above: $e^{i\sum\lambda_i(B_{t_i} - B_{t_{i-1}})} \cdot e^{\frac{1}{2}\sum\lambda_i^2(t_i - t_{i-1})} =$

$$\mathcal{E}_\infty^f = 1 + i\int_0^\infty \mathcal{E}_s^f f(s)\,dB_s$$

$\Rightarrow e^{i\sum\lambda_j(B_{t_j} - B_{t_{j-1}})} \in \mathcal{H}$. By lemma. $\mathcal{H}$ is dense.

<u>Cor.</u> i) $\forall$ mart. M bdd in L^2. Then $\exists$ unique $h \in L^2(B)$. const. $C \in \mathbb{R}^1$. s.t. $M_t = C + \int_0^t h_s\,dB_s$

ii) $\forall$ c.l.m M. Then $\exists$ unique $h \in L^2_{loc}(B)$ and const. $C \in \mathbb{R}^1$. s.t. $M_t = C + \int_0^t h_s\,dB_s$.

<u>Pf:</u> i) Apply on $M_\infty \in L^2(\Omega, \mathcal{F}_\infty, P)$

$M_\infty = E(M_\infty) + \int_0^\infty h_s\,dB_s$. $E(M_\infty | \mathcal{F}_t) = M_t$

ii) By Blumental Thm. $M_0 \equiv C$. since $M_0 \in \mathcal{F}_0$.

Set $T_n = \inf\{t \geq 0 \mid |M_t| \geq n\}$.

$\Rightarrow M^{T_n}$ satisfies the conditions in i).

$M_t^{T_n} = C + \int_0^t h_s^{(n)}\,dB_s$. By uniqueness:

$h_s^{(m)} = I_{[0, T_m]}(s) h_s^{(n)}$ for $m < n$. a.s. a.e.

Set $h_s I_{[0, T_m]} = h_s^{(m)}$. $\Rightarrow h_s \in L^2_{loc}(B)$.

<u>Thm.</u> (Multi dimensional Extension)

Suppose $(\mathcal{F}_t)$ is complete canonical filtration of a d-dim SBM $(\vec{B_t}) = (B_t^1, \cdots B_t^d)$

Then $\forall Z \in L^2(\Omega, \mathcal{F}_\infty, P)$. $\exists$ unique $\vec{h} = (h_1 \cdots h_d)$

s.t. $h_i \in L^2(B)$. $\forall 1 \le i \le d$. $Z = E(Z) + \sum_1^d \int_0^\infty h_s^i \, dB_s^i$

Consequencently. the identical results hold for

M. mart. bdd in L^2 or c.l.m.

Pf: Consider $\mathcal{E}_t^{\vec{f}} = \mathcal{E}(i \sum_1^d \int_0^\cdot f_k(s) \, dB_s^k)_t$

$f_k(s) = \sum_0^{n-1} I_{(t_i^k, t_{i+1}^k]}$ check $\mathcal{H}$ is still closed. dense.

(Note the prop. w.r.t $\mathcal{E}$ can be extended to

multidimension case: $\mathcal{E}(\vec{\lambda} \cdot \vec{M})$. $F(\vec{r}, \vec{x}) = e^{\sum_1^d \lambda_i x_i - \frac{\lambda_i^2}{2} r_i}$)

② Applications:

i) The filtration $(\mathcal{F}_t)$ is conti:

We have check $\mathcal{F}_t$. $\mathcal{F}_t^+$. $\mathcal{F}_t^-$ only differ a P-null

set when learning BM. Next. we use the represent

Thm to check again.

Pf: For $Z \in \mathcal{F}_t^+$. bdd $\exists h_s \in L^2(B)$. $Z = E(Z) + \int_0^\infty h \, dB$.

Note for $\varepsilon > 0$. $Z = E(Z | \mathcal{F}_{t+\varepsilon}) = E(Z) + \int_0^{t+\varepsilon} h \, dB$.

Set $\varepsilon \to 0$. RHS $\xrightarrow[a.s]{L^2} E(Z) + \int_0^t h_s \, dB_s \in \mathcal{F}_t$.

Since $(\mathcal{F}_t)$ is complete $\Rightarrow \mathcal{F}_t = \mathcal{F}_t^+$.

Conversely. for $Z \in \mathcal{F}_t$. $E(Z | \mathcal{F}_{t-\varepsilon}) = E(Z) + \int_0^{t-\varepsilon} h \, dB$

$Z = E(Z | \mathcal{F}_t) = E(Z) + \int_0^t h \, dB \nwarrow E(Z | \mathcal{F}_{t-\varepsilon})$. $\varepsilon \to 0$

$\Rightarrow Z = E(Z | \mathcal{F}_{t-})$. a.s. $\therefore \mathcal{F}_t = \mathcal{F}_t^-$. by complete.

ii) All marts w.r.t $(\mathcal{F}_t)$ has
a modification with conti paths:

For mart. M_t. WLOG. Suppose it's u.i.
Otherwise stop it at proper time. (M^n)
Then $\exists M_\infty$. $M_t = E(M_\infty | \mathcal{F}_t)$. $\forall t \geq 0$.
Note: $(\mathcal{F}_t)$ is conti. complete. $E(M_t)$ is conti
So M has a càdlàg modification $\tilde{M}$.
Set $M_\infty^{(n)} = \tilde{M}_\infty I_{\{|\tilde{M}_\infty| \leq n\}} + n I_{\{|\tilde{M}_\infty| > n\}}$. bdd
since $|M_\infty^{(n)}| \leq |\tilde{M}| \in L^1$. $\Rightarrow M_\infty^{(n)} \xrightarrow{L^1} \tilde{M}_\infty$ by DCT.
Set mart. $M_t^{(n)} = E(M_\infty^{(n)} | \mathcal{F}_t)$ bdd in L^2.
By represent Thm. $M^{(n)}$ are conti a.s.
Apply Doob's (Require right-conti. That's why use $\tilde{M}$!)
$\Rightarrow \forall \lambda > 0$. $P(\sup_t |M_t^{(n)} - \tilde{M}_t| > \lambda) \leq \frac{3}{\lambda} E|M_\infty^{(n)} - \tilde{M}_\infty| \to 0$
Select subseq (n_k). $\sup |M_t^{(n_k)} - \tilde{M}_t| \to 0$ a.s.
Since $\tilde{M}$ is uniform limit of conti. mart. a.s
$\Rightarrow \tilde{M}$ has conti modification. So does M.

(5) Girsanov's Thm:

Suppose filtration $(\mathcal{F}_t)$ is complete. right-conti
in $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ filtrated prob space.

① Thm:

prop. If Q is another p.m. on $(\Omega, \mathcal{F})$. $Q \ll P$ on $\mathcal{F}_\infty$.

$\forall t \in \bar{\mathbb{R}}^+$. Set $D_t = dQ/dP|_{\mathcal{F}_t}$. R-N derivate.

Then: (D_t) is u.i. mart.

Pf: $\forall A \in \mathcal{F}_t$. $Q(A) = E_Q(I_A) = E_P(I_A D_\infty) = E_P(I_A E(D_\infty|\mathcal{F}_t))$

$= E_P(D_t I_A)$. $\Rightarrow D_t = E(D_\infty|\mathcal{F}_t)$. a.s.

Cor. Under the conditions above. Then:

i) (D_t) has a càdlàg modification $(\tilde{D}_t)$.

ii) $\forall T$ stopping time. $D_T = dQ/dP|_{\mathcal{F}_T}$.

iii) if $P \ll Q$ on $\mathcal{F}_\infty$. then:

$\inf_{t \geq 0} \tilde{D}_t > 0$, $\sup_{t \geq 0} \tilde{D}_t < \infty$. P-a.s. Q-a.s.

Pf: i) Note $(\mathcal{F}_t)$ is complete, right-conti.

ii) By optional stopping Thm.

iii) $\forall \varepsilon > 0$. Set $T_\varepsilon = \inf\{t \geq 0 \mid \tilde{D}_t < \varepsilon\}$. stop. time

$\{T_\varepsilon < \infty\} \in \mathcal{F}_{T_\varepsilon} \Rightarrow Q(T_\varepsilon < \infty) = E_P(I_{\{T_\varepsilon < \infty\}} D_{T_\varepsilon}) \leq \varepsilon$

follows from right-conti of $\tilde{D}_t$.

$\Rightarrow Q(\cap\{T_{\frac{1}{n}} < \infty\}) = 0. \underset{P \ll Q}{\Rightarrow} P(\cap\{T_{\frac{1}{n}} < \infty\}) = 0$

By sym: $dP/dQ|_{\mathcal{F}_t} = 1/D_t$ holds as well.

prop. D is c.l.m. $D > 0$. P-a.s. Then. $\exists$ unique c.l.m L.

s.t. $D_t = \mathcal{E}(L)_t = e^{L_t - \frac{1}{2}\langle L, L\rangle_t}$. Actually. L has

form: $L_t = \log D_0 + \int_0^t \frac{dD_s}{D_s}$

Pf: uniqueness is trivial. Apply Itô's on $\log D_t$

follows from Rmk iv) of Itô's Thm.

Thm. (Girsanov's)

If $P \ll Q$. $Q \ll P$ on $\mathcal{F}_\infty$. $D_t = dQ/dP|_{\mathcal{F}_t}$ is a conti. mart. L is the unique c.l.m. s.t. $D_t = \mathcal{E}(L)_t$

Then: M is c.l.m under $P \Rightarrow \tilde{M} = M - \langle M, L\rangle$ is c.l.m under Q.

Lemma. X is conti. adapted process. T is stopping time. $(XD)^T$ is mart. under $P \Rightarrow X^T$ is mart. under Q.

Pf: $E_Q |X_{T\wedge t}| = E_P |X_{T\wedge t} D_{T\wedge t}| < \infty \Rightarrow X_t^T \in L^1(Q)$

$\forall A \in \mathcal{F}_s$. $s<t$. Then $A \cap \{T>s\} \in \mathcal{F}_s$.

$\Rightarrow E_P(I_{A\cap\{T>s\}} X_{T\wedge t} D_{T\wedge t}) = E_P(I_{A\cap\{T>s\}} X_{T\wedge s} D_{T\wedge s})$

Besides, $A \cap \{T>s\} \in \mathcal{F}_{T\wedge s} \subset \mathcal{F}_{T\wedge t}$

$\Rightarrow E_Q(I_{A\cap\{T>s\}} X_{T\wedge t}) = E_Q(I_{A\cap\{T>s\}} X_{T\wedge s})$

Combined with a trivial equation:

$E_Q(I_{A\cap\{T\le s\}} X_{T\wedge t}) = E_Q(I_{A\cap\{T\le s\}} X_{T\wedge s})$

Cor. (XD) is c.l.m under $P \Rightarrow X$ is c.l.m under Q.

Return to the Pf:

By Itô's : $\tilde{M}_t D_t = M_0 D_0 + \int_0^t \tilde{M}_s dD_s + \int_0^t D_s dM_s$

$- \int_0^t D_s d\langle M, L\rangle_s + \langle M, D\rangle_t$.

$= M_0 D_0 + \int_0^t \tilde{M}_s dD_s + \int_0^t D_s dM_s$.

$d\langle M, L\rangle_t = D_t^{-1} d\langle M, D\rangle_t$. by prop. above.

$\Rightarrow \tilde{M}D$ is c.l.m under P. Then by Lemma.

③ Consequence :

i) If $P \ll Q \ll P$. Then $P\text{-}\mathcal{M}^{conti}_{semi} = Q\text{-}\mathcal{M}^{conti}_{semi}$

<u>Pf:</u> 1°) Role of P. Q can be exchangable.

Consider $D_t^{-1} = dP/dQ|_{\mathcal{F}_t} = e^{\frac{1}{2}\langle L,L\rangle_t - L_t}$

Set $\tilde{L} = L - \langle L,L\rangle$ c.l.m on Q. $\Rightarrow D_t^{-1} = \mathcal{E}(-\tilde{L})_t$

Besides, $\langle \tilde{L},\tilde{L}\rangle = \langle L,L\rangle$.

2°) $\forall M$. c.l.m under P $\Rightarrow M = \tilde{M} + \langle M,L\rangle$. Semimart under Q.

$P\text{-}\mathcal{M}^{conti}_{semi} = \{P\text{-c.l.m} + FV\} = \{Q\text{-semimart} + FV\}$

$\subset \{Q\text{-semimarts}\} = Q\text{-}\mathcal{M}^{conti}_{semi}$

converly, by symmetry of P. Q.

<u>Rmk:</u> Set $\mathcal{G}^P_Q$: P-c.l.m's $\longrightarrow$ Q-c.l.m's

$M \longmapsto \tilde{M} = M - \langle M,L\rangle$

$\mathcal{G}^Q_P$: Q-c.l.m's $\longrightarrow$ P-c.l.m's

$M \longmapsto \tilde{M} = M - \langle M,-\tilde{L}\rangle$

$\Rightarrow \mathcal{G}^P_Q \circ \mathcal{G}^Q_P = \mathcal{G}^Q_P \circ \mathcal{G}^P_Q = Id.$

So : $\mathcal{G}^P_Q$, $\mathcal{G}^Q_P$ are bijections.

ii) Brackets of two conti semimarts X. Y are identical under P or Q. if $P \ll Q \ll P$.

<u>Pf</u>, By approxi. equation. Converge in p.m. $P \Leftrightarrow Q$. Since $P \sim Q$.

Moreover. for H locally bdd progressive process. The sto-integration $H \cdot X$ is same under P or Q.

Pf: Note that $X = M + A$. decomposition under P.

Since $H\cdot A =: \int_0^\cdot H_s dA_s$ irrelevant with choice of p.m. It's FV process under P or Q.

$\Rightarrow$ Prove: $(H\cdot M)_P = (H\cdot M)_Q$. M is c.l.m.

1°) Under P: $(H\cdot\widetilde{M})_P = (H\cdot M)_P - H\cdot\langle M, L\rangle$

$$= (H\cdot M)_P - \langle (H\cdot M)_P, L\rangle$$

$$= \widetilde{(H\cdot M)_P}$$. c.l.m. under Q.

2°) $\forall N$. c.l.m on Q. $\exists N'$ c.l.m. on P. s.t.

$\mathcal{G}_Q^P(N') = N$. i.e. $\widetilde{N'} = N$.

$$\langle (H\cdot\widetilde{M})_P, N\rangle = \langle \widetilde{(H\cdot M)_P}, \widetilde{N'}\rangle = \langle (H\cdot M)_P, N'\rangle$$

$$= H\cdot\langle M, N'\rangle = H\cdot\langle \widetilde{M}, \widetilde{N'}\rangle$$

$$= H\cdot\langle\widetilde{M}, N\rangle$$

$\Rightarrow$ By characterization. $(H\cdot\widetilde{M})_P = (H\cdot\widetilde{M})_Q$

So: By linearity. $(H\cdot M)_P = (H\cdot M)_Q$.

Rmk: From 1°). We have: $H\cdot\mathcal{G}_Q^P(M) = \mathcal{G}_Q^P(H\cdot M)$.

Cor. $\mathcal{G}_Q^P$. $\mathcal{G}_P^Q$ are isometric isomorphisms under H^2 norm. Since $\langle\widetilde{M}, \widetilde{M}\rangle_P = \langle M, M\rangle_P = \langle M, M\rangle_Q$.

iii) B is $(\mathcal{F}_t)$-BM under $P \overset{P\sim Q}{\Longrightarrow} \widetilde{B}$ is still $(\mathcal{F}_t)$-BM under Q.

Pf: $\langle B, B\rangle_t = \langle\widetilde{B}, \widetilde{B}\rangle_t = t$. By Lévy characterization.

③ Applications:

i) Construct P.m.:

Lemma. For M c.l.m Then. we have:

$\{\lim_{t\to\infty} M_t$ exists. finite$\} = \{\langle M,M\rangle_\infty < \infty\}$. a.s.

Pf: 1°) On $\{\langle M,M\rangle_\infty = \infty\}$. By represent in BM:

$$M_{\tau_t} = \beta_t . \quad \begin{cases} \overline{\lim}\, \beta_t = +\infty \\ \underline{\lim}\, \beta_t = -\infty \end{cases} . \quad \tau_t \underset{t\to\infty}{\nearrow} \infty .$$

$$\Rightarrow \begin{matrix} \overline{\lim_t}\, M_{\tau_t} = +\infty \le \overline{\lim}\, M_t \\ \underline{\lim_t}\, M_{\tau_t} = -\infty \ge \underline{\lim}\, M_t \end{matrix} . \quad \text{So: LHS} \subset \text{RHS}.$$

2°) Conversely. assume $M_0 = 0$. $T_n = \inf\{t \ge 0 \mid \langle M,M\rangle_t \ge n\}$

$\Rightarrow M^{T_n}$ is bdd in L^2. so $\lim_t M_t^{T_n}$ exists. a.s.

But on $\{\langle M,M\rangle_\infty = \infty\}$. a.s. $\exists n$. $T_n = \infty$. a.s.

Cor. For M. c.l.m. The followings equi.:

i) $\overline{\lim_{t\to\infty}}\, M_t = +\infty$. a.s. ii) $\underline{\lim_{t\to\infty}}\, M_t = -\infty$. a.s.

iii) $\langle M,M\rangle_\infty = \infty$. a.s.

Start from c.l.m L. $L_0 = 0$. $\langle L,L\rangle_\infty < \infty$. a.s.

By Lemma. $\lim_{t\to\infty} L_t$ exists. a.s.

$\mathcal{E}(L)_t$ is nonnegative c.l.m. So a supermart.

which converges to $\mathcal{E}(L)_\infty = e^{L_\infty - \frac{1}{2}\langle L,L\rangle_\infty}$.

By Fatou's: $E(\mathcal{E}(L)_\infty) \le 1$.

Claim: $E(\mathcal{E}(L)_\infty) = 1 \iff \mathcal{E}(L)$ is u.i. mart.

Pf: $(\Leftarrow)$ is trivial. For $(\Rightarrow)$:

Note: $E(\mathcal{E}(L)_\infty) = E(\mathcal{E}(L)_0) = E(\mathcal{E}(L)_t)$. $\forall t$.

Check: $E(\mathcal{E}(L)_\infty \mid \mathcal{F}_t) = \mathcal{E}(L)_t$. a.s.

follows from $\mathcal{E}(L)$ is supermart.

$\Rightarrow$ Set $dQ = \mathcal{E}(L)_\infty(\omega)\, dP(\omega)$. $D_t = \mathcal{E}(L)_t$. $\left(E_P\left(\frac{dQ}{dP}\Big|_{\mathcal{F}_t} I_A\right) = E_P(\mathcal{E}(L)_\infty I_A) \right)$

Next, we introduce a Thm ensuring Claim holds.

Thm. L is c.l.m. with $L_0 = 0$.

i) (Novikov's Criteria) $E(e^{\frac{1}{2}\langle L, L\rangle_\infty}) < \infty$

ii) (Kazamaki's Criteria) L is u.i. mart. $E(e^{\frac{1}{2}L_\infty}) < \infty$.

iii) $\mathcal{E}(L)$ is a u.i. mart.

Then: i) $\Rightarrow$ ii) $\Rightarrow$ iii).

Pf: i) $\Rightarrow$ ii): L is bdd in L^2 by $E(\langle L, L\rangle_\infty) < \infty$.

So it's u.i. mart.

Besides: $e^{\frac{1}{2}L_\infty} = (\mathcal{E}(L)_\infty)^{\frac{1}{2}} e^{\frac{1}{4}\langle L, L\rangle_\infty}$.

ii) $\Rightarrow$ iii): By Jensen inequi. $\forall T$ stopping time.

$e^{\frac{1}{2}L_T} \leq E(e^{\frac{1}{2}L_\infty} \mid \mathcal{F}_T) =: N_T$

N_T is u.i. since $e^{\frac{1}{2}L_\infty} \in L^1$.

$\Rightarrow e^{\frac{1}{2}L_T}$ is u.i. w.r.t stopping time T.

1) Next. prove: $\forall 0 < a < 1$. $\mathcal{E}(aL)_T$ is u.i. on

T. Stopping time.

Set $Z_t^{(a)} = e^{\frac{aL}{1+a}}$. $\mathcal{E}(aL) = \mathcal{E}(L)^{a^2} (Z_t^{(a)})^{1-a^2}$.

$\forall I \in \mathcal{F}$. By Hölder Inequi:

$$E(I_I \, \mathcal{E}(aL)_T) \le E(\mathcal{E}(L)_T)^{a^2} E(I_I \, Z_T^{(a)})^{1-a^2}$$

$$\overset{(\text{Jensen})}{\le} E(I_I \, e^{\frac{1}{2}L_T})^{2a(1-a)}$$

2) If T_n reduce $\mathcal{E}(aL)_t$. Then:

$E(\mathcal{E}(aL)_{t\wedge T_n} \mid \mathcal{F}_s) = \mathcal{E}(aL)_{s\wedge T_n}$. Let $n \to \infty$. by u.i.

So $\mathcal{E}(aL)_t$ is u.i. mart.

$\Rightarrow 1 = E(\mathcal{E}(aL)_\infty) \le E(\mathcal{E}(L)_\infty)^{a^2} E(e^{\frac{1}{2}L_\infty})^{2a(1-a)}$

Then let $a \to 1$. $E(\mathcal{E}(L)_\infty) \ge 1$.

ii) Construct Solution of SDEs:

If b is bdd measurable on $\mathbb{R}^+ \times \mathbb{R}^1$. $\exists g \in L^2(\mathbb{R}^+, B_{\mathbb{R}^+}, dt)$

s.t. $|b(t,x)| \le g(t)$. $\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^1$. B is $(\mathcal{F}_t)$-BM.

Consider $L_t = \int_0^t b(s, B_s)\, dB_s$, c.l.m.

It satisfies Novikov's criteria. $E(e^{\frac{1}{2}\langle L, L\rangle_\infty}) < \infty$.

$\Rightarrow D_t = \mathcal{E}(L)_t$ is u.i. mart. Set $Q = D_\infty \cdot P$.

By Girsanov Thm:

$\beta_t = B_t - \langle B, L\rangle_t = B_t - \int_0^t b(s, B_s)\, ds$ is $(\mathcal{F}_t)$-BM on Q.

Rmk: Restate these above: Under p.m. Q. $\exists (\mathcal{F}_t)$-BM β

s.t. $X = B$ solves SDE: $dX_t = d\beta_t + b(t, X_t)\, dt$.

iii) Cameron - Martin Formula:

Def: $\mathcal{H}$ is set of all func's $h(t)$. s.t. $\exists g \in L^2(\mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+}, dt)$, $\forall t \geq 0$

$h(t) = \int_0^t g(s)ds$, called Cameron-Martin space.

Rmk: By argument of ii). $Q = D_\infty \cdot P = e^{\int_0^\infty g(s)dB_s - \frac{1}{2}\int_0^\infty g^2 ds} \cdot P$

is p.m. s.t. $\beta_t = B_t - h(t)$ is BM under Q.

Prop. (Cameron-Martin Formula)

$W(dw)$ is Wiener measure on $C(\mathbb{R}^+, \mathbb{R})$, $h \in \mathcal{H}$.

Then. $\forall \phi$ nonnegative measurable on $C(\mathbb{R}^+, \mathbb{R})$.

$$\int \phi(w+h) W(dw) = \int \phi(w) e^{\int_0^\infty h'(s) dw(s) - \frac{1}{2}\int_0^\infty h'^2 ds} W(dw).$$

Pf: By Rmk above: $E_P(D_\infty \phi((B_t)_{t\geq 0})) = E_Q(\phi((B_t)_{t\geq 0}))$

$= E_Q(\phi((\beta_t + h(t))_{t\geq 0})) = E_P(\phi(B_t + h(t))_t)$.

Set $P = W$. (h' is weak derivate of h)

Rmk: C-M Formula gives kind of "quasi-invariant" property of Wiener measure under translation $h \in \mathcal{H}$.

i.e. $\theta_h : w \longmapsto w + h(t)$. $W \circ \theta_h^{-1}$ has density:

$$\frac{dW \circ \theta_h^{-1}}{dW}(w) = \mathcal{E}\left(\int_0^\cdot h(s) dW_s\right)_\infty$$

Set $U_a = \inf\{t \geq 0 \mid B_t + ct = a\}$. hitting time with drift.

Apply C-M Formula on $h'(s) = c\, I_{\{s \leq t\}}$, $h(s) = c(s \wedge t)$.

$\phi(w) = I_{\{\max_{0\leq s\leq t} W_s \geq a\}}$. Then: $P(U_a \leq t) = \int_0^t \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{(a-cs)^2}{2s}} ds$

$$\Rightarrow P(U_a < \infty) = \begin{cases} 1 & \text{if } c \geq 0 \\ e^{2ac} & \text{if } c \leq 0 \end{cases}$$