Local Times

Motivations: i) Find a method to measure the time of Bm B_t spends at "$x = a$".

Since: $E(\int I_{\{B_t = a\}} dt) = \int E(I_{\{B_t = a\}}) dt = 0$.

ii) Generalize Itô's Formular.

(1) Tanaka's Formula:

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. $(\mathcal{F}_t)$ is complete. X is conti. semimart.

① Convex Func's:

i) f'_+, f'_- exists. $\forall x \in \mathbb{R}'$. f'_+ is right-conti. f'_- is left-conti.

Pf: $\Delta f(x, h) = \dfrac{f(x+h) - f(x)}{h}$ is monotone on h.

ii) $f'_+(x) \leq f'_-(\eta) \leq f'_+(\eta)$. for $\forall x < \eta$.

iii) $\exists (\alpha_n), (\beta_n) \subset \mathbb{R}'$. s.t. $f(x) = \sup_n (\alpha_n x + \beta_n)$

Pf: By $f(\eta) \leq \dfrac{z-\eta}{z-x} f(x) + \dfrac{\eta - x}{z-x} f(z)$. $x < \eta < z$. Choose $\eta_n \to x$

iv) Convex func is conti. on $\forall$ open interval. $\left(\dfrac{f(x)-f(a)}{x-a} \leq \dfrac{f(b)-f(a)}{b-a} \right)$

v) $f'_+ = f'_-$, m-a.e. (m is Lebesgue measure). Moreover, f'_+, f'_- has the same set of continuities. and $f'_+ \neq f'_-$ only on a countable set.

vi) $\forall \varphi$. $f'_-(x) \leq \varphi(x) \leq f'_+(x) \Rightarrow f(t) = \int_0^t \varphi(x) dx + f(0)$.

vii) f'' may not exist for f convex function

Instead, we define second derivate measure

$f''(a,b] = f'_-(b) - f'_-(a)$. f_- is left-conti.

nondecreasing. So. $f''(dy)$ is its Radon measure.

Rmk: f'' is weak derivate of f'_-. (in sense of distribution function)

② Thm. (Tanaka's Formula)

f convex on $\mathbb{R}^1$. Then $\exists$ increasing process A^f. s.t.

$\forall t \geq 0$. $f(X_t) = f(X_0) + \int_0^t f'_-(X_s)\,dX_s + A_t^f$. P-a.s.

So $f(X_t)$ is also a conti semimart.

Pf: $\ell \geq 0$. (ℓ_n) is Mollifiers. Set $\varphi_n = \ell_n * f$

$\varphi_n \in C^\infty$. $\varphi_n' = \ell_n * f'_-$. φ_n is convex as well.

$\varphi_n \to f$. $\varphi_n' \to f'_-$. since $f \in C$. f'_- is left-conti

Set $T_k = \inf\{t \geq 0 \mid |X_t| + \langle M,M\rangle_t + \int_0^t |dV_s| \geq k\}$. $X = M + V$

Apply Itô's on $\varphi_n(X_{t\wedge T_k})$. Then by DCT:

$$\int_0^{t\wedge T_k} \varphi_n'(X_s)\,dX_s \xrightarrow[P]{n\to\infty} \int_0^{t\wedge T_k} f'_-(X_s)\,dX_s.$$

Set: $A_t^{f,k} = f(X_{t\wedge T_k}) - f(X_0) - \int_0^{t\wedge T_k} f'_-(X_s)\,dX_s$

which is limit of $\frac{1}{2}\int_0^{t\wedge T_k} \varphi_n''(X_s)\,d\langle M,M\rangle_s$ on n. in pr.

$(A_t^{f,k})_{t\geq 0}$ $\uparrow$ conti. Set $A_t^f : A_{t\wedge T_k}^f = A_t^{f,k}$

well-def since: $A_t^{f,k} = A_{t\wedge T_k}^{f,k'}$. $\forall k' \geq k$.

Rmk: Similarly. $\exists \tilde{A}_t^f$. increasing process. st.

$f(X_t) = f(X_0) + \int_0^t f'_+(X_s)\,dX_s + \tilde{A}_t^f$ But in general. $\tilde{A}_t^f \neq A_t^f$. (Note $f \in C^2 \Rightarrow \tilde{A}_t^f = A_t^f$)

<u>Denote</u>: For convention. set $sgn(x) = I_{\{x>0\}} - I_{\{x\le 0\}}$.

<u>prop.</u> $\forall a \in \mathbb{R}^1$. There exists $L_t^a(X)$ increasing process. st.

$$|X_t - a| = |X_0 - a| + \int_0^t sgn(X_s - a)\,dX_s + L_t^a(X)$$

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{\{X_s > a\}}\,dX_s + \frac{1}{2}L_t^a(X).$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t I_{\{X_s \le a\}}\,dX_s + \frac{1}{2}L_t^a(X).$$

Besides, $\forall T$ stopping time. $L_t^a(X^T) = L_{t\wedge T}^a(X)$.

<u>Pf</u>: Apply Tanaka's Formula on $f(x) = |x - a|$.

Then on $(X-a)^+$. $(X-a)^-$:

$$\begin{cases} (X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{\{X_s > a\}}\,dX_s + A_t^{a,+} \\ (X_t - a)^- = (X_0 - a)^- + \int_0^t I_{\{X_s \le a\}}\,dX_s + A_t^{a,-} \end{cases}$$

$$\Rightarrow \begin{cases} A_t^{a,+} = A_t^{a,-}. & \text{(take difference)} \\ A_t^{a,+} + A_t^{a,-} = L_t^a(X). & \text{(take sum)} \end{cases}$$

<u>Def</u>: $(L_t^a(X))_{t\ge 0}$ is called Local time of X at level a.

Denote $d_s L_s^a(X)$ is random measure of $s \mapsto L_s^a(X) \uparrow$.

i.e. $\int_0^t d_s L_s^a(X) = L_t^a(X)$.

<u>prop.</u> $\forall a \in \mathbb{R}^1$. Then. a.s. $supp(d_s L_s^a(X)) \subset \{s \ge 0 \mid X_s = a\}$.

<u>Pf</u>: Set $W_t = |X_t - a| \Rightarrow \langle W, W\rangle = \langle X, X\rangle$.

Apply Itô's on W_t^2. combined with Tanaka's on $|X_t - a|$

$$\Rightarrow (X_t - a)^2 = (X_0 - a)^2 + 2\int_0^t (X_s - a)\,dX_s + 2\int_0^t |X_s - a|\,d_s L_s^a(X) + \langle X, X\rangle_t$$

Compared with Itô on $(X_t-a)^2$.

$$\Rightarrow \int_0^t |X_s-a|\, ds L_s^a(X) = 0.$$

Rmk: It shows $t \mapsto L_t^a(X)$ only increases on $\{t\ge 0 \mid X_t=a\}$. So, in some sense. $L_t^a(X)$ can measure number of visits of X at a. before time t.

(2) Generalized Itô Formula:

① Continuity of Local Time:

Denote: Write $L^a(X)$ for random conti. func. $(L_t^a(X))_{t\ge 0}$ which is r.v. taking values in $C(\mathbb{R}^+, \mathbb{R}^+)$. equipped with topo. of u.c.c. convergence.

Lemma. $\forall p\ge 1$. $\exists C_p$. st. $\forall a<b\in\mathbb{R}'$. We have: for $X=M+V$

$$E\left(\left(\int_0^t I_{\{a<X_s<b\}}\, d\langle M,M\rangle_s\right)^p\right) \le C_p(b-a)^p\left(E(\langle M,M\rangle_t^{\frac{p}{2}}) + E\left(\left(\int_0^t |dV_s|\right)^p\right)\right)$$

Pf: 1°) WLOG. set $a=-u$. $b=u$. $u>0$.

by set $u=(b-a)/2$. replace X by $X-\frac{b+a}{2}$.

Set $f\in C^2(\mathbb{R})$. St. $f''(x)=(2-\frac{|x|}{u})^+$. $f'(0)=f(0)=0$

$\Rightarrow |f'(x)|\le 2u$. $\forall x$. $f''\ge 0$. $f''(x)\ge 1$. if $-u<x<u$.

So: $\int_0^t I_{\{-u<X_s<u\}}\, d\langle M,M\rangle_s \le \int_0^t f''(X_s)\, d\langle M,M\rangle_s$.

2) By Itô on f:

$$\frac{1}{2}\int_0^t f''(X_s)\, d\langle M,M\rangle_s = f(X_t)-f(X_0) - \int_0^t f'(X_s)\, dX_s$$

Apply C_r-inequility. separate $|f(X_t)-f(X_0)|^p$. $|\int_0^t \square|^p$.

3°) $E(|f(X_t)-f(X_0)|^p) \overset{|f'|\le 2n}{\le} (2n)^p E|X_t-X_0|^p$

$$\overset{Cr.\ BDG}{\le} C_p (2n)^p \left(E(\langle M,M\rangle_t^{\frac{p}{2}}) + E((\int_0^t |dV_s|)^p)\right)$$

4°) $\int_0^t f'(X_s)\,dX_s = \int_0^t f'(X_s)\,dM_s + \int_0^t f'(X_s)\,dV_s$.

By Cr-inequality and BDG inequality again:

$$E((\int_0^t f'(X_s)\,dV_s)^p) \le (2n)^p E((\int_0^t |dV_s|)^p)$$

$$E(|\int_0^t f'(X_s)\,dM_s|^p) \le C_p E((\int_0^t f'^2\,d\langle M,M\rangle_s)^{\frac{p}{2}})$$

$$\le (2n)^p C_p E(\langle M,M\rangle_t^{\frac{p}{2}})$$

RMK: Set $X = X^{T_n}$. Let $a \to b$. Then: $\forall b \in \mathbb{R}^1$:

$\int_0^t I_{\{X_s=b\}}\,d\langle M,M\rangle_s = 0$. $\forall t \ge 0$. a.s.

$\Rightarrow$ Also. $\int_0^t I_{\{X_s=b\}}\,dM_s = 0$. $\forall t \ge 0$. a.s.

Cor. $\forall a \in \mathbb{R}^1$. $Y_t^a = \int_0^t I_{\{X_s>a\}}\,dM_s$. $Y^a = (Y_t^a)_{t\ge 0}$ r.v. take values in $C(\mathbb{R}^+, \mathbb{R}^1)$. Then: $(Y^a)_{a\in\mathbb{R}^1}$ has a continuous modification.

Pf. Fix $p>2$. By BDG inequality:

$$E(\sup_{s\le t}|Y_s^b - Y_s^a|^p) \le C_p E((\int_0^t I_{\{a<X_s\le b\}}\,d\langle M,M\rangle_s)^{\frac{p}{2}})$$

Set $T_n = \inf\{t\ge 0 \mid \langle M,M\rangle_t + \int_0^t |dV_s| \ge n\}$.

$$E((\int_0^{t\wedge T_n} I_{\{a<X_s\le b\}}\,d\langle M,M\rangle_s)^{\frac{p}{2}}) \le C_p (b-a)^{\frac{p}{2}} (n^{\frac{p}{4}} + n^{\frac{p}{2}})$$

follows from the Lemma above.

Replace X by X^{T_n}. and set $t \to \infty$.

$$\Rightarrow E(\sup_{s\ge 0}|Y^b_{s\wedge T_n} - Y^a_{s\wedge T_n}|^p) \le C_p (n^{\frac{p}{4}} + n^{\frac{p}{2}})(b-a)^{\frac{p}{2}}$$

Then by Kolmogorov's Lemma. $a \mapsto (Y^a_{s\wedge T_n})_{s\ge 0}$

has a $(\frac{1}{2}-\frac{1}{p})$-Hölder Modification $(Y^{(n),a}_s)_{s\ge 0}$

Since: $Y^{(m),a}_s = Y^{(m),a}_{s\wedge T_n}$. $n<m$. $\forall s$. a.s.

$\Rightarrow$ Set $(\widetilde{Y}^a)$: $\widetilde{Y}^a_{s\wedge T_n} = Y^{a,(n)}_s$. a.s. $n\to\infty$.

Thm. $(L^a(X))_{a\in\mathbb{R}}$ with values in $(C(\mathbb{R}^+, \mathbb{R}^+)$ has a càdlàg modification $(\widetilde{L}^a(X))_{a\in\mathbb{R}}$. Denote $\widetilde{L}^{a-}(X) = (\widetilde{L}^{a-}_t(X))_{t\ge 0}$ the left limit of $s\mapsto \widetilde{L}^s(X)$ at $s=a$. Then:

$\forall t\ge 0$. $\widetilde{L}^a_t(X) - \widetilde{L}^{a-}_t(X) = 2\int_0^t I_{\{X_s=a\}}\, dV_s$. a.s.

Rmk: Note if X is c.l.m. then $V_s=0$. i.e.

$(\widetilde{L}^a_t(X))_{t\ge 0, a\in\mathbb{R}}$ has joint conti sample paths.

Pf: Set: $Z^a_t = \int_0^t I_{\{X_s>a\}}\, dV_s$. $Z^a = (Z^a_t)_{t\ge 0}$.

By Tanaka's on $(X_t-a)^+$:

$L^a_t = 2((X_t-a)^+ - (X_0-a)^+ - Y^a_t - Z^a_t)$. $\forall t\ge 0$. a.s.

By DCT. $a\mapsto Z^a$ has càdlàg paths.

Besides: $Z^{a_0}_t - Z^{a_0-}_t = -\int_0^t I_{\{X_s=a_0\}}\, dV_s$

Combined with the cor. above.

Rmk: Apply Tanaka's on $W_t = |X_t|$. $f(x) = x^+$. $W_t = W_t^+$.

$\Rightarrow W_t = |X_0| + \int_0^t I_{\{|X_s|>0\}}\, d|X_s| + \frac{1}{2}L^0_t(W)$

$= |X_0| + \int_0^t I_{\{|X_s|>0\}}(\mathrm{sgn}(X_s)\, dX_s + d_s L^0_s(X)) + \frac{1}{2}L^0_t(W)$

$= |X_0| + \int_0^t \mathrm{sgn}(X_s)\, dX_s + \int_0^t I_{\{X_s=0\}}\, dX_s + \frac{1}{2}L^0_t(W)$

since $\mathrm{supp}(d_s L^0_s(X)) \subset \{X_t=0\}$

Compare with Tanaka's on X_t. $f(x) = |x|$.

$\Rightarrow L_t^0(W) = 2L_t^0(X) - 2\int_0^t I_{\{X_s=0\}} dX_s = L_t^0(X) + L_t^{0-}(X)$.

More generally. $L_t^a(W) = L_t^a(X) + L_t^{a-}(X)$. $\forall a \in \mathbb{R}^1$.

② **Thm.** (Generalized Itô's Formula)

f is difference of two convex functions on $\mathbb{R}^1$.

Then. $\forall t \geq 0$. $f(X_t) = f(X_0) + \int_0^t f'_-(X_s) ds + \frac{1}{2}\int_{\mathbb{R}^1} L_t^a(X) f''(da)$ a.s.

Rmk: Note $L_t^a(X) = 0$. for $a \notin [\min_{s \leq t} X_s, \max_{s \leq t} X_s]$. a.s.

$\Rightarrow a \mapsto L_t^a(X)$ is bdd. So the term $\int_{\mathbb{R}} L_t^a(X) f''$

makes sense.

Pf: 1°) Consider f is convex by linear. WLOG:

suppose f'' supports on $[-k, k]$. by Rmk above.

and $f = 0$ on $[-\infty, -k]$. by translation.

Integrate by part: $\begin{cases} f'_-(x) = \int_{\mathbb{R}} I_{\{a<x\}} f''(da) \\ f(x) = \int_{\mathbb{R}} (x-a)^+ f''(da) \end{cases}$

From: $(X_t - a)^+ = (X_0 - a)^+ + Y_t^a + Z_t^a + \frac{1}{2} L_t^a(X)$

$\Rightarrow f(X_t) = f(X_0) + \int_{\mathbb{R}^1} Y_t^a f''(da) + \int_{\mathbb{R}^1} Z_t^a f''(da) + \square$

2°) By Fubini: $\int_{\mathbb{R}^1} Z_t^a f''(da) = \int_{\mathbb{R}} (\int_0^t I_{\{X_s > a\}} dV_s) f''(da)$

$= \int_0^t f'_-(X_s) dV_s$.

3°) prove: $\int_{\mathbb{R}^1} Y_t^a f''(da) = \int_0^t f'_-(X_s) dM_s$. a.s.

Set $T_n = \inf\{s \geq 0 \mid \langle M, M\rangle_s \geq n\}$

Consider Y_t^a is conti (by modify).

Since supp f'' is cpt. $\Rightarrow Y_t^a$ is bdd in $\int$.

So: WLOG. Consider Y^{T_n}, then let $n \to \infty$.

prove: $\int \left(\int_0^{t\wedge T_n} I_{\{X_s > a\}} \, dM_s \right) f''(da) \triangleq M_t^f =$

$$\int_0^{t\wedge T_n} \left(\int I_{\{X_s > a\}} f''(da) \right) dM_s \triangleq \tilde{M}_t^f$$

Note: M_t^f, $\tilde{M}_t^f \in H^2$. Then $\forall N \in H^2$:

$E(\langle M^f, N\rangle_\infty) = E \int \left(\int_0^{T_n} I_{\{X_s > a\}} \, d\langle M, N\rangle_s \right) f''(da))$

$$\overset{\text{Fubini}}{=} E\left(\int_0^{T_n} \left(\int I_{\{X_s > a\}} f''(da) \right) d\langle M, N\rangle_s \right)$$

$$= E(\langle \tilde{M}^f, N\rangle_\infty)$$

follows from $\langle \, , \, \rangle$ commutes with $\int$.

<u>Cor.</u> (Density of occupation time Formula)

$\int_0^t \varphi(X_s) \, d\langle X, X\rangle_s = \int_{\mathbb{R}^1} \varphi(a) L_t^a(X) \, da$. $\forall t \geq 0$

$\forall \varphi \geq 0$ measurable on $\mathbb{R}^1$. P-a.s.

More generally, $\int_0^\infty F(s, X_s) \, d\langle X, X\rangle_s =$

$\int_{\mathbb{R}^1} \left(\int_0^\infty F(s, a) \, ds L_s^a(X) \right) da$. $\forall F \geq 0$ and

measurable on $\mathbb{R}_+ \times \mathbb{R}^1$. P-a.s.

<u>Pf:</u> 1°) Set $f \in C^2$, s.t. $f'' = \varphi \geq 0$.

So f is convex.

Compare the equations from Itô's

on f and "Generalized Itô's" on f

$\Rightarrow \int_0^t \varphi(X_s) \, d\langle X, X\rangle_s = \int_{\mathbb{R}^1} \varphi(a) L_t^a(X) \, da$.

We only consider $\varphi \in$ countable dense

Set in $C_c(\mathbb{R}^1)$ to retain a.s. holds

2') First, it holds for $F(s,a) = I_{[u,v]}(s) I_A(a)$ where $0 \le u < v$, $A \in \mathcal{B}_{\mathbb{R}^1}$ by 1')

$\Rightarrow$ So for $F(s,a) = I_B(s,a)$ by MCT.

Cor. If M has form: $M_t = M_0 + V_t$, V_t is FV process. Then: $L_t^a(M) = 0$. $\forall a \in \mathbb{R}^1$, $t \ge 0$.

Pf: By occupation time formula:

$\int_{\mathbb{R}} \varphi(a) L_t^a(M) da = 0$ by $\langle M, M\rangle_t = 0$. $\forall t$

for $\forall \varphi \ge 0$. measurable. So: $L_t^a(M) = 0$. a.s.

(3) Approxi of Local Times:

① prop. X is conti. semimart. Then. a.s. $\forall a \in \mathbb{R}^1$. $\forall t \ge 0$.

$$L_t^a(X) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t I_{[a \le X_s \le a+\varepsilon]} d\langle X, X\rangle_s.$$

Pf: By Formula: $\frac{1}{\varepsilon} \int_0^t I_{[a \le X_s \le a+\varepsilon)} d\langle X, X\rangle_s = \frac{1}{\varepsilon} \int_a^{a+\varepsilon} L_t^b(X) db$

$\xrightarrow{\varepsilon \to 0} L_t^a(X)$. since $b \mapsto L_t^b(X)$ is right-conti.

Cor. $\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t I_{[a-\varepsilon \le X_s \le a+\varepsilon]} d\langle X, X\rangle_s = \frac{1}{2}(L_t^a(X) + L_t^{a-}(X))$

Rmk: $\tilde{L}_t^a(X) =: \frac{1}{2}(L_t^a(X) + L_t^{a-}(X))$ is called symmetric local time of semimart. X.

Actually. $\tilde{L}_t^a$ only differs at most countable points from L_t^a. (nondecreasing)

$\Rightarrow$ Occupation time Formula remains true if replace $L_t^a(X)$ by $\tilde{L}_t^a(X)$

Besides. replace f'_- by $\frac{1}{2}(f'_- + f'_+)$

in Itô's and Tanaka's Formulas if

consider on $\tilde{L}^a_t(X)$.

Cor. $p \geq 1$. $\exists C_p$. s.t. $\forall$ conti. semimart. $X = M + V$

$\forall a \in \mathbb{R}^1$. $t \geq 0$. We have:

$$E((L^a_t(X))^p) \leq C_p(E(\langle M, M\rangle_t^{\frac{p}{2}}) + E((\int_0^t |dV_s|)^p))$$

Pf: By approxi. of local time. Fatou's Lemma.

② Upcrossing approxi of local time:

For X conti semimart. $\varepsilon > 0$.

Def: i) $\sigma^\varepsilon_{n+1} = \inf\{t \geq \tau^\varepsilon_n \mid X_t = 0\}$, $\sigma^\varepsilon_1 = \inf\{t \geq 0 \mid X_t = 0\}$

ii) $\tau^\varepsilon_{n+1} = \inf\{t \geq \sigma^\varepsilon_n \mid X_t = \varepsilon\}$, $\tau^\varepsilon_1 = \inf\{t \geq \sigma^\varepsilon_1 \mid X_t = \varepsilon\}$.

iii) Upcrossing number of X along $[0, \varepsilon]$ before time t

is $N^X_\varepsilon(t) =: \#\{n \geq 1 \mid \tau^\varepsilon_n \leq t\}$.

prop. $\forall t \geq 0$. $\varepsilon N^X_\varepsilon(t) \xrightarrow[P]{\varepsilon \to 0} \frac{1}{2} L^0_t(X)$

Pf: 1°) Apply Tanaka's on X. $f(x) = x^+$:

$$(X_{\tau^\varepsilon_n \wedge t})^+ - (X_{\sigma^\varepsilon_n \wedge t})^+ = \int_{\sigma^\varepsilon_n \wedge t}^{\tau^\varepsilon_n \wedge t} I_{\{X_s > 0\}} dX_s + \frac{1}{2}(L^0_{\tau^\varepsilon_n \wedge t} - L^0_{\sigma^\varepsilon_n \wedge t})$$

By DCT: $\sum_{n=1}^\infty ((X_{\tau^\varepsilon_n \wedge t})^+ - (X_{\sigma^\varepsilon_n \wedge t})^+) =$

$$\int_0^t (\sum_n I_{(\sigma^\varepsilon_n, \tau^\varepsilon_n]}) I_{\{X_s > 0\}} dX_s + \frac{1}{2}\sum_{n=1}^\infty (L^0_{\tau^\varepsilon_n \wedge t} - L^0_{\sigma^\varepsilon_n \wedge t})$$

2°) L^0_t will not increase on $[\tau^\varepsilon_n, \sigma^\varepsilon_{n+1})$ $\forall n$. $\tau^\varepsilon_0 \triangleq 0$

$$\Rightarrow \sum (L^0_{\tau_n^\varepsilon \wedge t} - L^0_{\sigma_n^\varepsilon \wedge t}) = \sum (L^0_{\sigma_{n+1}^\varepsilon \wedge t} - L^0_{\sigma_n^\varepsilon \wedge t})$$

$$= L^0_t.$$

3°) combined with $(X_{\tau_n^\varepsilon \wedge t})^+ - (X_{\sigma_n^\varepsilon \wedge t})^+ = \varepsilon$. if $\tau_n^\varepsilon \le t$.

$\Rightarrow$ LHS $= \varepsilon N_\varepsilon^X(t) + u(\varepsilon)$. $0 \le u(t) \le \varepsilon \xrightarrow{\varepsilon \to 0} 0$

4°) $\left(\sum_{n=1}^{\infty} I_{(\sigma_n^\varepsilon, \tau_n^\varepsilon]}(s)\right) I_{\{X_s > 0\}} \le I_{\{0 < X_s \le \varepsilon\}} \xrightarrow{\varepsilon \to 0} 0$

Then apply DCT. the last term $\xrightarrow[p]{\varepsilon \to 0} 0$

(4) Local Times of linear BM:

(B_t) is one-dim real SBM. $(\mathcal{F}_t)$ is its complete filtration.

Thm. (Trotter's)

There exists a unique process $(L_t^a(B))_{t \ge 0, a \in \mathbb{R}^1}$ whose sample paths are conti on (a, t). St. $(L_t^a(B))_{t \ge 0}$ is increasing process. $\forall a$ fixed. and a.s.: $\forall a \in \mathbb{R}^1$. $\operatorname{supp}(d_s L_s^a(B)) \subset \{t \ge 0 \mid B_t = a\}$.

If fix $a \in \mathbb{R}^1$. $\operatorname{supp}(d_s L_s^a(B)) = \{t \ge 0 \mid B_t = a\}$. a.s.

Rmk: $L_s^a(B)$ can exactly measure the number of B_t visiting level a.

Pf: The former is trivial. since B is c.l.m.

1°) Consider $a \in \mathbb{Q}$. $\operatorname{supp}(d_s L_s^a(B)) \subset \{t \ge 0 \mid B_t = a\}$ holds a.s. Then by conti. argument:

$\Rightarrow$ extend to $\forall a \in \mathbb{R}^1$.

Pf: if $\{\exists a \in \mathbb{R}^1.\ 0 \le s < t.\ L_t^a(B) > L_s^a(B)$.

$B_r \neq a.\ \forall r \in [s,t]\}$ has prob >0.

Then we can find $b \in \mathbb{Q}$ closed to a.

s.t. $L_t^b(B) > L_t^b(B)$. $B_r \neq b.\ \forall r \in [s,t]$.

since $B.\ L_t^a(B)$ are conti. Contradict!

(*) Recall that $x \in \mathrm{supp}(\mu)$ μ is p.m. $\Leftrightarrow \forall I.$ open nbd of x. $\mu(I) > 0$.

2') Fix $a \in \mathbb{R}'$. Set $H_q = \inf\{t \geq q \mid B_t = a\}.\ \forall q \in \mathbb{Q}$.

(Since $\forall t.$ s.t. $B_t = a.\ \exists q_n \uparrow t.\ q_n \in \mathbb{Q}$)

prove: a.s: for $\forall \varepsilon > 0.\ L_{H_q}^a(B) > L_{H_q - \varepsilon}^a(B)$. (*)

By Strong Markov Property at $H_q - \varepsilon$:

prove: $L_\varepsilon^a(B') > 0$. for Bm. B'_t. $\forall \varepsilon > 0$. a.s.

WLOG. $a = 0$. Note: $L_\varepsilon^0(B) \overset{d}{\sim} \sqrt{\varepsilon}\, L_1^0(B)$.

From $E(L_1^0(B)) = E(|B_1|)$ by Tanaka's

$\Rightarrow P(L_1^0(B) > 0) = P(L_{2^{-n}}^0(B) > 0) > 0$

Apply Blumental's: $A = \cap\{L_{2^{-n}}^0(B) > 0\} \in \mathcal{F}_0^+$

$P(A) = \lim\limits_n P(L_{2^{-n}}^0(B) > 0) > 0 \Rightarrow P(A) = 1$.

Similarly. a.s: $\forall \varepsilon > 0.\ L_{H_q+\varepsilon}^a(B) > L_{H_q}^a(B)$

Rmk: It remains true for arbitrary B_0.

Prop. (distribution property)

i) $a \in \mathbb{R}'/\{0\}$. $T_a = \inf\{t \geq 0 \mid B_t = a\}$ Then: $L_{T_a}^0(B) \sim \mathrm{Exp}(2|a|)$

ii) $a > 0$. $U_a = \inf\{t \geq 0 \mid |B_t| = a\}$. Then: $L_{U_a}^0(B) \sim \mathrm{Exp}(a)$.

Pf: i) 1') WLOG. by scaling. sym. set $a = 1$

Note: $L_\infty^0(B) = \infty$. a.s. Since $L_\infty^0(B) \sim \lambda L_\infty^0(B)$

by scaling. and $\exists \varepsilon.\ P(L_\infty^0 > \varepsilon) = P(L_\infty^0 > \frac{\varepsilon}{\lambda}) = 1.\ \forall \lambda$.

Fix $s>0$. Set $\tau = \inf\{t\geq 0 \mid L_t^0(B)\geq s\}$. $B_\tau = 0$.

By strong Markov property: $B_t' =: B_{t+\tau}$ is BM.

which is indept with $\mathcal{F}_\tau$.

$L_t^0(B') = \lim \frac{1}{\varepsilon}\int_0^t I_{\{0\leq B_{\tau+s}\leq \varepsilon\}}\,ds = \lim \frac{1}{\varepsilon}\int_0^t (L^{\cdot}_{\tau+t}(B) - L^{\cdot}_{\tau}(B))\,da$

$= L^0_{\tau+t}(B) - s$.

On $\{L^0_{T_1}(B)\geq s\} = \{\tau\leq T_1\}$. $\Rightarrow L^0_{T_1}(B) - s = L^{\cdot}_{T_1-\tau}(B') = L_{T_1'}(B')$

where $T_1' = \inf\{t\geq 0 \mid B_t' = 1\}$. indept with $\mathcal{F}_\tau$.

$\Rightarrow P(L^{\cdot}_{T_1}(B) - s\geq t \mid L^0_{T_1}(B)\geq s) = P(L^0_{T_1'}(B')\geq t \mid L^0_{T_1}(B)\geq s)$

$= P(L^0_{T_1}(B)\geq t)$

from $\{L^0_{T_1}(B)\geq s\} = \{\tau\leq T_1\}\in\mathcal{F}_\tau$. So it's exp. dist.

2') By $|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_s)\,dB_s + L_t^\cdot(B)$. (Tanaka's)

$<\int_0^\cdot \operatorname{sgn}(B_s)\,dB_s, \int_0^\cdot \operatorname{sgn}(B_s)\,dB_s>_t = t$. $E(B_t^+) = E(B_t^-)$

$\Rightarrow E(B^+_{t\wedge T_1}) = \frac{1}{2}E(L^\cdot_{t\wedge T_1})$. Let $t\to\infty$ by DCT. MCT.

So: $E(L^0_{T_1}(B)) = 2$.

ii) It's identical. with $E(L^0_{t\wedge T_1}(B)) = E(|B_{t\wedge T_1}|)$.

Denote: $S_t =: \sup_{[0,t]} B_s$. $I_s = \inf_{[0,t]} B_s$. $\beta_t =: -\int_0^t \operatorname{sgn}(B_s)\,dB_s$.

Lemma. $L_t^0(B) = \sup_{s\leq t}\beta_s$ for $\forall t\geq 0$

Pf: $L_t^0(B)\geq L_s^0(B) = |B_s| + \beta_s\geq\beta_s$. $\forall s\leq t$.

Conversely. Set $Y_t = \sup\{s<t \mid B_s = 0\}$. So: $L^0_{Y_t}(B) = L^0_t(B)$

$\Rightarrow L^0_{Y_t}(B) = \beta_{Y_t}\leq\sup\{\beta_s \mid s\leq t\}$.

Thm. (Lévy)

$(S_t, S_t - B_t) \overset{d}{\sim} (-I_t, B_t - I_t) \overset{d}{\sim} (L_t^\cdot(B), |B_t|)$. $\forall t\geq 0$.

Pf: The first "$\sim$" is by symmetry.

Note β_t is SBm we have proved before. Then:

$$(L_t^0(B), |B_t|) \overset{a.s.}{=} (\sup_{[0,t]} \beta_s, \sup_{[0,t]} \beta_s - \beta_t) \overset{d}{\sim} (S_t, S_t - B_t)$$

Cor. $L_t^0(B) \sim S_t \sim |B_t|$. $\forall t \geq 0$.

Def: $\tau_s = \inf\{t \geq 0 \mid L_t^0(B) > s\}$. $\forall s \geq 0$. inverse local time at level 0 of Bm $(B_t)_{t\geq 0}$.

Prop. i) $(\tau_s)_{s\geq 0}$ is increasing càdlàg.

ii) $(\tau_s)_{s\geq 0} \overset{d}{\sim} (\tilde{T}_s)_{s\geq 0}$. where $\tilde{T}_s = \inf\{t \geq 0 \mid B_t > s\}$.

iii) $(\tau_s)_{s\geq 0}$ has stationary indept. increments.

Rmk: These say $(\tau_s)_{s\geq 0}$ is a stable subordinator with index $\frac{1}{2}$

(Subordinator is a nondecreasing Lévy process)

Prop. $\{t \geq 0 \mid B_t = 0\} = \{\tau_s\}_{s\geq 0} \cup \{\tau_{s-}\}_{s\in D}$ a.s. D is countable set of jump times of $(\tau_s)_{s\geq 0}$.

Pf: 1) $\mathrm{supp}(ds L_s^0(B)) = \{t \geq 0 \mid B_t = 0\}$. a.s. $\tau_s, \tau_{s-} \in \mathrm{supp}(ds L_s^0)$.

Since $\mathrm{supp}(ds L_s^0(B))$ is closed. $\tau_{t_n} \uparrow \tau_{s-}$ if $s \in D$.

2) If $B_t = 0$. Then either $L_{t+\varepsilon}^0(B) > L_t^0(B)$. $\forall \varepsilon > 0$.

So $t = \inf\{s \geq 0 \mid L_s^0(B) > L_t^0(B)\} = \tau_{L_t^0(B)} \swarrow \tau_s$. $s \downarrow L_t^0$.

Or L is const. on $[t, t+\varepsilon]$. $L_t^0(B) > L_{t-\delta}^0(B)$. $\forall 0<\delta<t$.

So $t = \tau_{L_t^0(B)-} \nwarrow \tau_s$. $(s \uparrow L_t^0(B))$

Rmk: $Z(B)^c = \bigcup_{s\in D} (\tau_{s-}, \tau_s)$ union of connected components. which are called excursion intervals.