Markov Process

(1) Existence:

① **Def:** i) $(E, \mathcal{E})$ is a measurable space. A Markovian transition kernel from E into E is a map $Q: E \times \mathcal{E} \longrightarrow [0,1]$. satisfies:

(a) $\forall x \in E$. $A \in \mathcal{E} \longmapsto Q(x, A)$ is a p.m on E.

(b) $\forall A \in \mathcal{E}$. $x \in E \longmapsto Q(x, A)$ is $\mathcal{E}$-measurable.

Rmk: When E is countable. equipped $\mathcal{E} = P(E)$

Then Q is charac. by matrix $(Q(x, \{y\}))_{E \times E}$.

ii) For $f: E \to \mathbb{R}'$. bdd. measurable. $Qf(x) = \int_E Q(x, dy) f(y)$

iii) $(Q_t)_{t \ge 0}$ transition kernels on E is called a transition Semigroup if:

(a) $\forall x \in E$. $Q_0(x, dy) = \delta_x(dy)$

(b) $\forall s, t \ge 0$. $A \in \mathcal{E}$. $Q_{t+s}(x, A) = \int_E Q_t(x, dy) Q_s(y, A)$.

(c) $\forall A \in \mathcal{E}$. $(t, x) \longmapsto Q_t(x, A)$ is $B_{\mathbb{R}^+} \otimes \mathcal{E}$ - measurable.

Rmk: i) (b) $\Leftrightarrow$ $Q_{t+s} = Q_t Q_s$

ii) $(Q_t)_{t \ge 0}$ is collection of contractions on $B(E) = \{ f: E \to \mathbb{R}' \mid f \text{ bdd. measurable} \}$ equipped with norm $\|f\| = \sup_E |f(E)|$. which is a linear space.

iv) A Markov process w.r.t $(\mathcal{F}_t)_{t\geq 0}$ with transition semigroup $(Q_t)_{t\geq 0}$ is $(\mathcal{F}_t)$ - adapted process $(X_t)_{t\geq 0}$. s.t. $\forall s, t \geq 0$. $f \in B(E)$

$$E(f(X_{s+t}) | \mathcal{F}_s) = Q_t f(X_s)$$

Denote: $\mathcal{F}_t^X = \sigma(X_r, 0 \leq r \leq t)$

Rmk: Set $f = I_A$, $A \in \mathcal{E}$. Then we have:

$P(X_{s+t} \in A | \mathcal{F}_s^X) = Q_t(X_s, A)$. it's Markov property. i.e. Future depends on Present.

prop. For $0 = t_0 < t_1 \cdots < t_p$. $A_0 \cdots A_p \in \Sigma$. $f_0 \cdots f_p \in B(E)$. $X_0 \overset{d}{\sim} \gamma$

We have: $E(\prod_0^p f_i(X_{t_i})) = \int_{A_0} \gamma(dx_0) f_0(x_0) \cdots \int_{A_p} Q_{t_p - t_{p-1}}(x_p, dx_{p-1}) f(x_p)$

Pf. Set $I_{E_i} = f_i$. holds by def.

Then apply MCT argument.

Rmk. A Markov Process is completely determined by $(Q_t)_{t\geq 0}$ and the law of X_0.

e.g. $E = \mathbb{R}^d$. $Q_t(x, dy) = p_t(y-x)dy$. $p_t(z) = (2\pi t)^{-\frac{d}{2}} e^{-|z|^2/2t}$

$(Q_t)_{t\geq 0}$ is semigroup of d-dim SBM $(B_t)_{t\geq 0}$.

② Construction:

Set $\Omega^* = E^{\mathbb{R}^+} = \{w : \mathbb{R}^+ \to E\}$. equipped with $\mathcal{F}^* = \sigma(w \mapsto w(t), t \in \mathbb{R}^+, w \in \Omega^*)$. Let $(X(t))_{t\geq 0}$ be the canonical process on Ω^*. i.e. $X_t(w) = w(t)$. $t \geq 0$.

$X_t : \Omega^* \to E$.

Thm. For E is polish space. $(Q_t)_{t\geq 0}$ is transition semigroup on E. γ is p.m. on E. Then exists a unique p.m P on Ω^*. s.t. $(X_t)_{t\geq 0}$ is Markov process with transition semigroup $(Q_t)_{t\geq 0}$. $X_0 \sim \gamma$

Pf: $\forall\, u = \{t_i\}_1^p$. $0 \leq t_1 \cdots < t_p$. Define P^u on E^u:

$$\int P^u(dx_1 \cdots dx_p) I_A(x_1 \cdots x_p) = \int \gamma(dx_0) \cdots \int Q_{t_p - t_{p-1}}(x_{p-1}, dx_p) I_A(x_1 \cdots x_p).$$

for $\forall A \in \bigotimes_u \mathcal{E}$

It's easy to check it satisfies consistent condition in Kolmogorov Extention Thm. follows from $Q_{t+s} = Q_t Q_s$.

Rmk: Denote P_x is p.m. in Thm with $\gamma = \delta_x$.

$x \mapsto P_x(A)$ is $\mathcal{E}$-measurable. $\forall A \in \mathcal{F}^*$.

For any p.m. μ on E. define:

$P_{(\mu)}(A) = \int \mu(dx) P_x(A) \Rightarrow X_0 \sim \mu$ on $(E, P_{(\mu)})$

(3) **Resolvent:**

Note that $(Q_t)_{t\geq 0}$ is contraction on $B(E)$..

Def: $\forall \lambda > 0$. λ-resolvent of $(Q_t)_{t\geq 0}$ transition semigroup is $R_\lambda : B(E) \to B(E)$. $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} Q_t f(x) dt$.

$\forall f \in B(E)$. $x \in E$. linear operator.

Rmk: i) $\|R_\lambda\| \leq 1/\lambda$

ii) If $0 \leq f \leq 1$. Then: $0 \leq \lambda R_\lambda f \leq 1$.

Lemma: X is Markov Process with $(Q_t)_{t\geq 0}$ w.r.t $(\mathcal{F}_t)_{t\geq 0}$

$h \geq 0 \in B(E)$, $\lambda > 0$. Then: $e^{-\lambda t} R_\lambda h(X_t)$ is a

$(\mathcal{F}_t)$ — supermart.

Pf: Note: $e^{-\lambda t} R_\lambda h(X_t)$ is bdd.

$$Q_s R_\lambda h(X_r) = \int_0^\infty e^{-\lambda t} Q_{s+t} h(X_r) dt.$$

$$\Rightarrow e^{-\lambda s} Q_s R_\lambda h \leq R_\lambda h$$

So: $E(e^{-\lambda(t+s)} R_\lambda(h(X_{t+s})) | \mathcal{F}_t) =$

$$e^{-\lambda(t+s)} Q_s R_\lambda h(X_t) \leq e^{-\lambda t} R_\lambda h(X_t).$$

(2) Feller Semigroups:

Assume E is metrizable, locally cpt, σ-cpt topo Space equipped with Borel σ-field. (So E is polish)

Suppose $E = \cup K_n$, union of cpt sets, $K_n \uparrow E$.

① Def: i) $C_0(E) = \{ f \in C(E, \mathbb{R}) \mid \sup_{E/K_n} |f| \xrightarrow{n\to\infty} 0 \}$, equipped

with $\|f\| = \sup_E |f|$.

Rmk: i) $C_0(E)$ is a Banach space (Algebra)

ii) $C_0(E)^* \simeq M_m^{\mathbb{R}}$

iii) $C_0(E)$ can be approxi. by Stone-Weierstrass.

That's why we will consider $C_0(E)$.

ii) Trans. semigroup $(Q_t)_{t\geq 0}$ on E is feller semigroup if it's C_0- semigroup. i.e. satisfies:

(a) $\forall f \in C_0(E)$. $Q_t f \in C_0(E)$. $\forall t \geq 0$.

(b) $\forall f \in C_0(E)$. $\|Q_t f - f\| \to 0$. as $t \downarrow 0$.

<u>Denote</u>: L is infestimal generator of $(Q_t)_{t\geq 0}$.

Set: (b^*) $\forall f \in C_0(E)$. $|Q_t f(x) - f(x)| \xrightarrow{t \downarrow 0} 0$. $\forall x \geq 0$.

<u>Lemma</u>. For $(Q_t)_{t\geq 0}$ satisfies (a), (b^*). Then:

i) $R(R_\lambda)$ doesn't depend on choice of λ.

ii) $\mathcal{R} = \{ R_\lambda f \mid f \in C_0(E) \}$ is dense in $C_0(E)$.

<u>Pf</u>: i) By Resolvent equation: $R_{\lambda_1} f = R_{\lambda_2}(f + (\lambda_2 - \lambda_1) R_{\lambda_1} f)$

ii) By DCT $\Rightarrow R(R_\lambda) \subset C_0(E)$.

$\forall f \in C_0(E)$. $\lambda R_\lambda f(x) = \int_0^{+\infty} e^{-t} P_{t/\lambda} f(x) dt \underset{DCT}{\xrightarrow{\lambda \to \infty}} f(x)$

but it holds only pointwise $x \in E$.

consider $f^* \in C_0(E)^*$. Vanishes on $R(R_\lambda)$

By Riesz Representation: $\exists\, \mu$ radon measure.

s.t. $\langle f^*, f \rangle = \int_E f d\mu \quad \mu(S) = \|f^*\|$.

$0 = \int_E \lambda R_\lambda f(x) \mu(dx) = \int_E \int_0^\infty e^{-r} P_{r/\lambda} f(x) dr d\mu(x)$

$\xrightarrow{\lambda \to \infty} \int_E f(x) \mu(dx)$. by DCT.

$\Rightarrow \mu$ is zero measure. So $f^* = 0$.

<u>Rmk</u>: Note $D(L) = \mathcal{R}$ by $(\lambda - L) R_\lambda = id$.

prop. For $(Q_t)_{t\geq 0}$ trans. semigroup. Satisfies (a). (b*).

Then $(Q_t)_{t\geq 0}$ is Feller semigroup.

Pf: $P_t R_\lambda f(x) = P_t \int_0^\infty e^{-\lambda s} P_s f(x)\, ds$ (By Fubini)

$$= e^{\lambda t} \int_t^\infty e^{-\lambda r} P_r f(x)\, dr.$$

$$\Rightarrow |P_t R_\lambda f(x) - R_\lambda f(x)| = |(e^{\lambda t} - 1) \int_0^\infty e^{-\lambda r} P_r f(x)\, dr - e^{\lambda t} \int_0^t e^{-\lambda r} P_r f(x)\, dr|$$

$$\leq |e^{\lambda t} - 1| \, \|R_\lambda f\| + t e^{\lambda t} \|f\|$$

$$\to 0 \quad \text{as } t \downarrow 0.$$

which is indept with x. Then by denseness.

Thm. (Connect with C_0-semigroup)

For $(T_t)_{t\geq 0}$ C_0. contraction. positive. If:

$(f_n) \subset C_0(E)$. $f_n \uparrow 1$ pointwise $\Rightarrow T_t f_n \to 1$ pointwise

Then $\exists$ unique transtion semigroup $(Q_t)_{t\geq 0}$.

s.t. $T_t f(x) = \int_E f(y) Q_t(x, dy)$. $\forall f \in C_0(E)$.

Lemma. X is Banach. $\mathcal{A}$ is infestimal generator of some strongly conti semigroup of contraction on X. with domain $D(\mathcal{A})$ If G is extension of $\mathcal{A}$. st. $Gx = x \Rightarrow x = 0$. $\forall x \in D(G)$. Then:

$G = \mathcal{A}$ on $D(G)$.

Pf: $x \in D(G)$. Set $\eta = x - Gx$. $z = R_1\eta \in D(G)$

$\Rightarrow z - Gz = (I - G)R_1\eta = \eta = x - Gx$.

So: $G(x-z) = x - z$. $x = z \in D(G)$.

e.g. (Real Brownian Motion, d-dimension)

i) Semigroup $(Q_t)_{t\geq 0}$ of BM $(B_t)_{t\geq 0}$ is Feller

Pf: $|Q_t f(x) - f(x)| \leq \frac{1}{\sqrt{2\pi t}}\left(\int_{\mathbb{R}^d/B(x,\delta)} + \int_{B(x,\delta)} |f(y) - f(x)| e^{-\frac{|y-x|^2}{2t}} dy\right)$

$\leq \frac{2\|f\|}{\sqrt{2\pi t}} \int_{\mathbb{R}^d/B(x,\delta)} e^{-|y-x|^2/2t} dy + \varepsilon$.

$= C\|f\| \int_{\mathbb{R}^d/B(0,\delta/\sqrt{t})} e^{-|z|^2} dz + \varepsilon$

$\rightarrow 0$. as $t \rightarrow 0$. indept of x.

ii) $R_\lambda f(x) = \int \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x-y|} f(y) dy$. $\forall f \in C_0(E)$.

Pf: Note $X_t = e^{uB_t - \frac{1}{2}u^2 t}$ is mart. $E(X_{T_b \wedge t}) = E(X_0)$

by DCT. Let $t \to \infty$. $\therefore E(X_{T_b}) = 1$.

$\Rightarrow E(e^{-\lambda T_b}) = e^{-b\sqrt{2\lambda}}$. Differentiate on λ:

$E(T_b e^{-\lambda T_b}) = \frac{b}{\sqrt{2\lambda}} e^{-b/\sqrt{2\lambda}}$. By density of T_b:

$\Rightarrow \int_0^\infty t e^{-\lambda t} \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/2t} dt = \frac{b}{\sqrt{2\lambda}} e^{-b/\sqrt{2\lambda}}$

Set $b = |x - y|$. Simplify $R_\lambda f(x)$.

iii) $D(L) = \{h \in C^2(\mathbb{R}^1).\ h, h'' \in C_0(\mathbb{R}^1)\}$ when $d = 1$

$D(L) \subsetneq \{h \in C^2(\mathbb{R}^d).\ h, h'' \in C_0(\mathbb{R}^d)\}$ when $d \geq 2$.

Pf: Set $\lambda = \frac{1}{2}$. $h = R_{\frac{1}{2}} f$.

$$h'(x) = \int \mathrm{sgn}(\eta - x) e^{-|\eta - x|} f(\eta) d\eta.$$

Check: $h'(x) - h'(x_0) / (x - x_0) \xrightarrow{x \to x_0} -2f(x_0) + h(x_0)$

i.e. $h'' = -2f + h$ exists.

Combined with $(\frac{1}{2} - L) h = f \Rightarrow Lh = h''$. $h \in D(L)$

$\Rightarrow D(L) \subset \{h \in C^2, h, h'' \in C_0(\mathbb{R}^d)\}$

For $d=1$. $G = d^2/dx^2$ is LO extends L.

$Gf = f'' = f$. $f \in D(L) \Rightarrow f = 0$. Since $f \in C_0$

By Lemma. $G = L$.

Rmk: i) Note generator is determined locally.

So most case Lf only depend on the property of f only in nbd of x.

But in some case. it's global:

e.g. Cauchy (1) process:

$$Lf(x) = \frac{1}{\pi} \int_{\mathbb{R}^1} \frac{f(x+\eta) - f(x) - f'(x)\phi(\eta)}{\eta^2} d\eta$$

ii) Lf at most involve first. second order of f. high order derivate won't appear.

e.g. $Lf = f'''$. doesn't hold for generator L.

Pf: Choose $f(x) = \cos x + \frac{x}{2}$ on $[-2, 2]$.

$$Lf(\tfrac{2}{6}) = \lim_{t \to 0} \frac{Q_t f - f(\frac{2}{6})}{t} \leq 0. \text{ Contradict!}$$

Thm. $c(x) \in C_0(\mathbb{R}^1)$. $\|c\| \leq k$. (X_t) is Feller Process

s.t. $Q f(x) = c(x) f''(x)$. Q is its generator.

$\forall f \in C_0^2(\mathbb{R}^1)$. Then $(X_t)_{t\geq 0}$ is diffusion process.

② Mart. Problem:

Def: $D(M, E) = \{ f: M \to E,\ f \text{ is càdlàg} \}$. is called Skorokhood Space. Denote: $D(\mathbb{R}^{\geq 0}, E) = D(E)$

Next. we consider on $(\Omega, \mathcal{F}) = (D(E), D(E) \cap \mathcal{F}^*)$

$(X_t)_{t\geq 0} \in D(E) \cap \mathcal{F}^*$ adapted $(\mathcal{F}_t)_{t\geq 0}$ with $(\theta_t)_{t\geq 0}$

Thm. $h, q \in C_0(E)$. Then following equi.:

i) $h \in D(L)$ and $Lh = q$

(*)

ii) $\forall x \in E$. $h(X_t) - \int_0^t q(X_s) ds$ is mart. w.r.t $(\mathcal{F}_t)_{t\geq 0}$ under p.m. P_x.

Pf: i) $\Rightarrow$ ii) $\forall s \geq 0$. $Q_t h = h + \int_0^t Q_s q\, ds$.

$$E(h(X_{s+t}) | \mathcal{F}_t) = Q_s h(X_t)$$
$$= h(X_t) + \int_0^s Q_r q(X_t) dr$$

$$E\left(\int_t^{t+s} q(X_r) dr \,\middle|\, \mathcal{F}_t\right) = \int_t^{t+s} E(q(X_r) | \mathcal{F}_t) dr$$
$$= \int_0^s Q_r q(X_t) dr$$

combine these two equations.

(*) Calculate $[f(X_\cdot)]_t$:

$f^2(X_t) - [f^2(X_0) + \int_0^t \mathcal{L} f(X_t)]$ is a mart.

$\Rightarrow$ By Charac.:

$[f(X_\cdot)]_t = f^2(X_0) + \int_0^t \mathcal{L} f(X_s) ds$

ii) $\Rightarrow$ i): $E(h(X_t) - \int_0^t g(X_s)ds) = h(x)$.

$= Q_t h(x) - \int_0^t Q_r g(x) dr$. by def of (Q_t)

$\Rightarrow Q_t h - h / t \to g \in C_0(E)$. $\therefore Lh = g$.

<u>prop.</u> $\forall P$. p.m on $(\Omega, \mathcal{G})$. s.t. $P(X_0 = x) = 1$. for some $x \in E$. Q is unbdd LO. If $M_t = f(X_t) - \int_0^t Qf(X_s)ds$ is mart. under P. $\forall f \in D(Q)$. Then $P = P_x$.

P_x correspond to p.m. s.t. X_t is Markov Process starts at x.

<u>Rmk:</u> From this prop. rather than Hille-Yosida Thm. We can construct semigroups $(P_t)_{t \geq 0}$ for appropriate Q. (By Stroock. Varadhan)

1') "Find P_x. s.t. $(X_t)_{t \geq 0}$ is Markov Process from Q" is our target.

2') Select (Q_n): $Q_n \to Q$. We have already Q_n correspond $P_x^{(n)} \sim (X_t^{(n)})_{t \geq 0}$

3') $P_x^{(n)} \to \tilde{P}$. $\tilde{P}$ is P_x what we need. Since M_t is still mart. under $\tilde{P}$.

<u>Pf:</u> For $g \in C_0(E)$. $\lambda > 0$. Set $f = R_\lambda(Q) g$

From $E(M_t | \mathcal{G}_s) = M_s$. multiply $\lambda e^{-\lambda t}$. $t \geq s$.

$\Rightarrow f(X_s) = E(\int_0^\infty e^{-\lambda t} g(X_{s+t}) dt \mid \mathcal{G}_s)$

Set $s = 0$. So: $f(x) = E(\int_0^\infty e^{-\lambda t} g(X_t) dt)$

By Thm. above. M_t is mart under P_x as well

$\Rightarrow \bar{E}(f(X_0)) = \bar{E}_x(f(X_0)) = E_x(\int_0^\infty e^{-\lambda t} g(X_t) dt)$

$= E(\int_0^\infty e^{-\lambda t} g(X_t) dt)$

By Fubini: $\int_0^\infty e^{-\lambda t} E(g(X_t)) dt = \int_0^\infty e^{-\lambda t} E_x(g(X_t))$

$\Rightarrow \bar{E}(g(X_t)) = E_x(g(X_t))$. $\forall g \in C_0(E)$.

So X_t has same dist. under P or P_x.

Then it's easy to check: $P(X_{t_i} \in A_i, 1 \le i \le n) = P_x(\square)$.

(3) Regularity of Sample Paths:

① Def: i) Stochastic Process $X = (X_t)_{t \ge 0}$ is quasi-left-conti.

if (T_n) seq of stopping times $\nearrow T \Rightarrow X_{T_n} \xrightarrow{a.s.} X_T$

as $n \to \infty$ on $\{T < \infty\}$.

Rmk: Left conti $\Rightarrow$ quasi-left-conti. But the converse is false. e.g. Homogenous Poisson Process on $\mathbb{R}_{\ge 0}$.

ii) T is random time. It's called predictable if there exists increasing stopping times (T_n) w.r.t $(\mathcal{F}_t)$. s.t. $T = \lim_n T_n$. and $T_n < T$. $\forall n$. a.s on $\{T < \infty\}$

Rmk: T is a stopping time:

$\{T \le t\} = \bigcap \{T_n \le t\} \in \mathcal{F}_t$. by def.

iii) Stopping time T w.r.t $(\mathcal{F}_t)$ is totally inaccessible if $P(T = S, T < \infty) = 0$. $\forall S$. predictable time of $(\mathcal{F}_t)$

Lemma. If T is totally inaccessible stopping time of $(\mathcal{F}_t)_{t\geq 0}$. T_n stopping times of $(\mathcal{F}_t) \uparrow T$ on $\{T<\infty\}$. Then: $P((\bigcap_{n\geq 1}\{T_n<T\}) \cap \{T<\infty\})=0$

Pf: Denote $A_n=\{T_n<T\}$. $A=\bigcap A_n \in \mathcal{F}_T$.

T^A is still stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$

$T_n^{A_n}$ is stopping time of $(\mathcal{F}_t)_{t\geq 0}$. $\forall n\geq 1$.

Note: $A_n \downarrow \Rightarrow T_n^{A_n} \uparrow T^A$. $T_n^{A_n}(w) < T^A(w)$. $w\in A$.

$S_0 = T^A$ is predictable $\Rightarrow P(T^A=T.\ T<\infty)=0$

Thm. $(X_t)_{t\geq 0}$ is right-conti Markov Process adapted to $(\mathcal{F}_t)_{t\geq 0}$. Then following are equi.:

i) X is quasi-left-conti.

ii) $\forall$ predictable stopping time T of $(\mathcal{F}_t)_{t\geq 0}$.
$X_{T-}=X_T$ a.s. on $\{T<\infty\}$.

iii) If $X_T\neq X_{T-}$ a.s. on $\{T<\infty\}$. for stopping time T
Then: T is totally inaccessible.

Pf: i) $\Rightarrow$ ii) $\exists T_n \uparrow T$. $T_n<\infty$. on $\{T<\infty\}$. a.s. $\forall n$.

$\Rightarrow \lim_n X_{T_n} = X_{T-} = X_T$ a.s. on $\{T<\infty\}$.

ii) $\Rightarrow$ iii) For T. s.t. $X_T\neq X_{T-}$. S. predictable time

On $\{T=S,\ T<\infty\}$: $X_T=X_S=X_{S-}=X_{T-}$. a.s.

$\Rightarrow P(T=S.\ T<\infty)=0$

iii) $\Rightarrow$ i): $\forall (T_n)\uparrow T$. increasing seq of stopping time.

On $\{X_{T-} = X_T,\ T<\infty\}$. $\lim X_{T_n} = X_T$ obviously

On $\{X_{T-} \neq X_T,\ T<\infty\} =: A$. Note X is progressive.

$\Rightarrow A \in \mathcal{F}_T$. T^A is totally accessible by iii)

Since $\lim T_n = T = T^A$ on $\{T^A<\infty\} = A$ By Lemma:

$\Rightarrow P(\bigcup\{T_n \geq \hat{T}\} \cup \{\hat{T}=\infty\}) = 1$. i.e. $\exists N.\ n\geq N.\ T_n = \hat{T}$ a.s. on A.

$\therefore$ On A: $\lim_n X_{T_n} = X_{T^A} = X_T$ a.s.

<u>Cor.</u> Homogeneous Poisson Process on $\mathbb{R}^{\geq 0}$ is quasi-left-conti.

<u>Pf:</u> $(N_t)_{t\geq 0}$ adapted $(\mathcal{F}_t)_{t\geq 0}$ with intensity λ.

Set T predictable. $T_n \uparrow T$.

Note $M_t = N_t - \lambda t$ is mart. Apply OST:

$E(N_{T\wedge t} - N_{(T\wedge t)-};\ T<\infty) = \lim_n E(M_{T\wedge t} - M_{T_n\wedge t};\ T<\infty)$

$= 0$

By MCT. Let $t\to\infty$. $\Rightarrow N_T = N_{T-}$ on $\{T<\infty\}$.

2 <u>Thm.</u> $(X_t)_{t\geq 0}$ is Markov Process with semigroup $(Q_t)_{t\geq 0}$

w.r.t. $(\mathcal{F}_t)$. take value in E. separable. metrizable. LC

Denote $\mathcal{N} = \{A\in\mathcal{F}_\infty \mid P(A)=0\}$. $\tilde{\mathcal{F}}_t = \mathcal{F}_t^+ \vee \sigma(\mathcal{N})$.

Then. $(X_t)_{t\geq 0}$ has a modification $(\tilde{X}_t)$ adapted $(\tilde{\mathcal{F}}_t)$

s.t. i) $(\tilde{X}_t)$ is càdlàg.

ii) $(\tilde{X}_t)$ is Markov Process with (Q_t) w.r.t $(\tilde{\mathcal{F}}_t)$.

iii) $(\tilde{X}_t)$ is quasi-left-conti.

<u>Pf:</u> 1°) By one-point-compactification: $E_\Delta = E\cup\{\Delta\}$.

Set $\tilde{f}(\Delta)=0$. for $f\in C_0(E)$. $\Rightarrow \tilde{f}\in C_0(E_\Delta)$.

Set $C_0^+(E) = \{f \in C_0(E) \mid f \geq 0\}$.

$\mathcal{H} = \{R_p f_n \mid p \in \mathbb{Z}_{\geq 1},\ n \in \mathbb{Z}_{\geq 0}\}$. where $(f_n) \subset C_0^+(E)$

is seq of func. separating in E_Δ. (cpt. metrizable)

So $\mathcal{H}$ is also separating in E_Δ ($\|pR_pf - f\| \to 0$)

2°) By Lemma before. $e^{-pt} h(X_t)$ is supermart.

Remark D is dense countable in $\mathbb{R}_{\geq 0}$.

$N_h^c \subset \Omega$. is set of $w \in \Omega$. s.t. $s \in D \mapsto e^{-ps} h(X_s)$

make finite upcrossings along $[a,b]$. $\forall a,b \in \mathbb{Q}^+$.

$N = \bigcup_{h \in \mathcal{H}} N_h$ is P-null. On N^c: X_{t+}, X_{t-} exists.

follows from $\mathcal{H}$ is separating. in E_Δ.

$$\text{set } \widetilde{X}_t(w) = \begin{cases} 0. & w \in N \\ \lim\limits_{s \in D \downarrow t} X_s(w). = X_{t+}(w). & w \in \Omega / N \end{cases}$$

$\Rightarrow \widetilde{X}_t \in \widetilde{\mathcal{F}}_t$. is E_Δ-càdlàg. since $h(\widetilde{X}_t)$ is. h is separating

3°) Show $P(X_t = \widetilde{X}_t) = 1$

Let. $f, g \in C_0(E)$. $(t_n) \in D \downarrow t$.

$$E(f(X_t) g(\widetilde{X}_t)) = \lim_n E(f(X_t) g(\widetilde{X}_{t_n}))$$

$$= \lim_n E(f(X_t) Q_{t_n - t}(X_t)) \overset{DCT}{=} E(f(X_t) g(X_t))$$

$\Rightarrow$ Approxi. $f, g \in C_B(E)$. So $(X_t, \widetilde{X}_t) \overset{d}{\sim} (X_t, X_t)$

4°) Prove ii): $\Leftrightarrow E(I_A f(\widetilde{X}_{s+t})) = E(I_A Q_t f(\widetilde{X}_s))$. $\forall A \in \widetilde{\mathcal{F}}_s$

$\Leftrightarrow E(I_A f(X_{s+t})) = E(I_A Q_t f(X_s))$. $A \in \mathcal{F}_{s+}$

Let $s_n \downarrow s$. $s_n \in D$. $s_n \leq s+t$.

$E(I_A f(X_{s+t})) = E(I_A Q_{s+t-s_n} f(X_{s_n}))$.

Let $n \to \infty$. by DCT.

5°) Prove: $t \mapsto \widetilde{X}_t(\omega)$ is càdlàg as E-valued.

($\widetilde{X}_{t-} \in E$ may not hold in a.s. sense)

Fix $g > 0 \in C_0^+(E)$, $x \in E$. $h = R_1 g > 0$. $\forall x \in E$ as well.

Set: $Y_t = e^{-t} h(\widetilde{X}_t) \geq 0$ Supermart. w.r.t. $(\widetilde{\mathcal{G}}_t)$. Càdlàg

$T^{(n)} = \inf\{t \geq 0 \mid Y_t < \frac{1}{n}\}$. $\nearrow T$. Stopping time.

$\Rightarrow$ Prove: $P(T < \infty) = 0$.

(Then: $\forall t \in [0, T^{(n)})$. $\widetilde{X}_t$, $\widetilde{X}_{t-} \in E$. because:

$Y_t > \frac{1}{n} > 0 \Rightarrow h(\widetilde{X}_t), h(\widetilde{X}_{t-}) > 0$. $h(\Delta) = 0$ and

redefine $\widetilde{X}_t(\omega) = x_0$ (fix) $\in E$ on $\{T < \infty\}$)

By Y is supermart. right-conti. ≥ 0. by OST:

$E(Y_{T+q} I_{\{T<\infty\}}) \leq E(Y_{T^{(n)}} I_{\{T^{(n)}<\infty\}}) \leq \frac{1}{n} \to 0$. $q \in \mathbb{Q}^+$

$\Rightarrow Y_{T+q} = 0$. a.s. on $\{T < \infty\}$. i.e. $Y_t = 0$ on $[T, \infty)$

follows from right-conti. on $\{T < \infty\}$.

Note. $\forall k \in \mathbb{Z}^+$. $Y_k > 0$. a.s. $\therefore P(T < \infty) = 0$.

6°) For iii): prove: $E(f(X_T) g(X_{T-})) = E(f(X_{T-}) g(X_{T-}))$

for $\forall f, g \in C_0(E)$.

Rmk: i) The point is using that nonnegative supermart

$e^{-\lambda t} h(X_t)$ to imply càdlàg property.

ii) Given $(X_t)_{t \geq 0}$ with $(P_x)_{x \in E}$.

Set $\widetilde{\mathcal{G}}_t = \mathcal{G}_t^+ \vee \sigma(\mathcal{N}')$. $\mathcal{N}' = \{A \in \mathcal{G}_\infty \mid P_x(A) = 0$

for every $x \in E\}$. By: $P_x(N_h) = 0$. $\forall x \in E$. $\forall h \in \mathcal{H}$.

So: $P_x(N) = 0$. $\forall x \in E$. Still $(N \in \mathcal{N}')$

By identical argument: $(\widetilde{X}_t)_{t \geq 0}$ is càdlàg modification of $(X_t)_{t \geq 0}$. w.r.t $(\widetilde{\mathcal{G}}_t)_{t \geq 0}$. under P_x. $\forall x \in E$

(4) Markov Property:

Next. we consider $(X_t)_{t\geq 0}$ is càdlàg under P_x. $\forall x$.

On $(D(E).\ D(E)\cap \mathcal{F}^*.\ (\mathcal{F}_t)_{t\geq 0}.\ P_x)$

Thm. (Simple Markov)

$\phi: D(E) \to \mathbb{R}_+$. measurable. For $(Y_t)_{t\geq 0}$

Markov Process with Semigroup $(Q_t)_{t\geq 0}$.

Then: $E(\phi((Y_{s+t})_{t\geq 0}) \mid \mathcal{F}_s) = E_{Y_s}(\phi((Y_t)_{t\geq 0}))$.

Pf: For $A = \{ f \in D(E) \mid f(t_i) \in B_i.\ 1\leq i\leq p \}$. $B_i \in \mathcal{E}$.

prove it holds for I_A. Then by MCT.

More generally. if $\varphi_i \in B(E)$. $1\leq i\leq p$.

By induction on p: ($p=1$ trivial)

$E(\varphi_1(Y_{s+t_1}) \cdots \varphi_p(Y_{s+t_p}) \mid \mathcal{F}_s) =$

$E(\prod_1^{p-1} \varphi_i(Y_{s+t_i})\ Q_{t_p - t_{p-1}}(Y_{s+t_{p-1}}) \mid \mathcal{F}_s) =$

$\int Q_{t_1}(Y_s, dx_1)\varphi_1(x_1) \cdots \int Q_{t_p - t_{p-1}}(x_{p-1}, dx_p)\varphi_p(x_p)$

Thm: $\phi: D(E) \to \mathbb{R}_+$. measurable. For $(Y_t)_{t\geq 0}$

Feller process with $(Q_t)_{t\geq 0}$. T is stopping

time w.r.t. $(\mathcal{F}_t^+)_{t\geq 0}$. Then $\forall x \in E$.

we have: $E(I_{\{T<\infty\}}\ \phi\circ\theta_T \mid \mathcal{F}_T) = I_{\{T<\infty\}} E_{Y_T}(\phi)$

Pf: 1) On $\{T<\infty\}$. $E_{Y_T}(\phi) \in \mathcal{F}_T$.

2) Show: $E(I_{A\cap\{T<\infty\}}\ \phi\circ\theta_T) = E(I_{A\cap\{T<\infty\}} E_{Y_T}(\phi))$

$\forall A \in \mathcal{F}_T$.

Similarly, consider $\varphi_i \in B(E)$, $1 \le i \le p$. By induction:

prove: $E(I_{\{T<\infty\}\cap A}\,\varphi_1(Y_{T+t_1})\cdots\varphi_p(Y_{T+t_p})) = E_{Y_T}(I_{\square} \int Q_{t_1}(Y_T, dX_1)\cdots)$

So, it suffices to prove: "$p=1$" case.

Set $T_n = \dfrac{[2^n T]+1}{2^n} \downarrow T$. (T_n) seq of stopping time.

$$E(I_{\{T<\infty\}\cap A}\,\varphi(Y_{T+t})) = \lim_n \sum_i E(I_{A\cap\{\frac{i-1}{2^n}\le T<\frac{i}{2^n}\}}\,\varphi(Y_{i/2^n+t}))$$

$$= \lim_n E(I_{A\cap\{T<\infty\}}\, Q_t(Y_{T_n}))$$

$$= E(I_{\{T<\infty\}\cap A}\, Q_t\varphi(Y_T)).$$

follows from conti. of Feller semigroup.

__Rmk:__ For discrete time Markov process, it satisfies both simple and strong Markov Property.

But in conti. time, a Markov process which is not Feller may not satisfy strong Markov.

__e.g.__ (X_t) starts at X. $p(X=0)=p(X=1)=\frac{1}{2}$.

$$\begin{cases} \text{If } X=0, \text{ then } X_t=0,\ \forall t>0. \\ \text{If } X\neq 0, \text{ then } X_t \sim \text{Bm. start at } 1. \end{cases}$$

Consider $T=T_0$.

It satisfies simple but not strong Markov.

(5) Classes of Feller Process:

① Feller Jump process:

__Def:__ i) Jump process is stochastic process having discrete movements (jumps)

ii) Markov Jump process is a jump process $(X_t)_{t\geq 0}$ with trans. semigroup $(Q_t)_{t\geq 0}$ which is Markov Process w.r.t. $(\mathcal{F}_t)_{t\geq 0}$

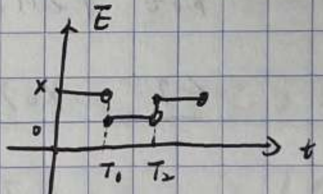
<u>Rmk</u>: State space E of Jump process can be conti. <u>e.g.</u> Compound Poisson process

Next. We consider $(X_t)_{t\geq 0}$ is Markov Jump process with Feller semigroup $(Q_t)_{t\geq 0}$. càdlàg. taking values in $\bar{E}$. at most countable. equipped with discrete topology. Consider in $(D(E), \mathcal{D}, P_x)$

Note: $\exists (T_n)$. $T_0(\omega) = 0 < T_1(\omega) \leq \cdots \leq T_n(\omega) \leq \infty$. s.t.

$X_t(\omega) = X_{T_i}(\omega)$. if $t \in [T_i(\omega), T_{i+1}(\omega))$. and

$X_{T_{i-1}} \neq X_{T_i}$. $\forall i \geq 1$.



<u>Rmk</u>: (T_n) is seq of stopping times: (induction)

$\{T_1 < t\} = \bigcup_{q\in[0,t)\cap\mathbb{Q}} \{X_q \neq X_0\}$. $\{T_2 < t\} = \bigcup_{a<b\in\mathbb{Q}\cap[0,t)} \{T_1 < a < b < T_2 < t\} \cup$

$\{T_2 = T_1 < t\}$.

<u>Lemma</u>: $x \in E$. $\exists\, q(x) \geq 0$. s.t. $T_1 \sim Exp(q(x))$ under P_x. Besides. $q(x) > 0 \Rightarrow T_1$. X_{T_1} indept under P_x.

<u>Pf</u>: 1°) $P_x(T_1 > s+t) = E_x(I_{\{T_1>s\}} \cdot I_{\{X_r = X_0,\ \forall r\in[0,t]\}} \circ \theta_s)$

$= P_x(T_1 > s)\, P_x(T_1 > t)$

follows from simple Markov Property

2°) For $q(x) > 0$. Then $T_1 < \infty$. P_x-a.s.

Similarly. $P_x(T_1 > t, X_{T_1} = \eta) = E_x(I_{\{T_1 > t\}}\, \varphi \circ \theta_t)$

$\varphi = I_{\{Y_1(X_t) = \eta\}}$. $Y_1(f)$ is first jump of f.

Apply simple Markov: $= P_x(T_1 > t)\, P_x(X_{T_1} = \eta)$.

<u>Rmk</u>: These holds for general Markov Jump Process.

<u>Denote</u>: $\pi(x, \eta) = P_x(X_{T_1} = \eta)$. for $x \in E$. $q(x) > 0$.

<u>Rmk</u>: It's a p.m. on E.

<u>prop.</u> L is generator of $(Q_t)_{t \geq 0}$. If $\sup\limits_{\eta} q(\eta) < \infty$. Then:

$B(E) \subseteq D(L)$. and $\forall\, \varphi \in B(E)$. $\forall x \in E$:

i). $q(x) = 0 \Rightarrow L\varphi(x) = 0$.

ii) $q(x) \neq 0 \Rightarrow L\varphi(x) = q(x) \sum\limits_{\eta \neq x} \pi(x, \eta)(\varphi(\eta) - \varphi(x))$

$$= \sum_{\eta \in E} L(x, \eta)\, \varphi(\eta).$$

where $L(x, \eta) = \begin{cases} q(x)\, \pi(x, \eta). & x \neq \eta \\ -q(x) & x = \eta \end{cases}$

<u>Pf</u>: i) $q(x) = 0. \Rightarrow Q_t \varphi(x) = \varphi(x)$. $\forall t \geq 0$.

ii) 1°) Prove: $P_x(T_2 \leq t) = O(t^2)$ $(t \to 0)$

$LHS \leq P(T_1 \leq t,\ T_2 - T_1 \leq t) = E_x(I_{\{T_1 \leq t\}}\, P_{X_{T_1}}(T_2 \leq t))$

by Strong Markov Property.

Note: $P_{X_{T_1}}(T_2 \leq t) \leq \sup\limits_{\eta} P_\eta(T_1 \leq \eta) \leq t \sup\limits_{\eta} q(\eta)$

$P_x(T_1 \leq t) \leq t \sup\limits_{\eta} q(\eta)$. Combine them.

2°) By $Q_t \varphi(x) = E_x(\varphi(X_t)) = E_x(\varphi(X_t) I_{\{t < T_1\}}) +$

$E_x(\varphi(X_{T_1}) I_{\{T_1 \leq t\}}) + O(t^2)$.

$$= \varphi(x) e^{-q(x)t} + (1 - e^{-q(x)t}) \sum_{\eta \neq x} \pi(x,\eta)\varphi(\eta) + O(t^2).$$

$$\Rightarrow Q_t\varphi(x) - \varphi(x) / t \to \sum L(x,\eta)\varphi(\eta)$$

<u>Rmk:</u> i) If $|E| < \infty$. Then $C_0(E) = B(E) = D(L)$.

ii) Set $\varphi(\eta) = I_{\{\eta\}}$. $\Rightarrow \frac{d}{dt} P_x(X_t = \eta)|_{t=0} = L(x,\eta)$.

for $x \neq \eta$. L is like $p'(0)$ in CTMC.

<u>Prop.</u> Suppose $q(\eta) > 0$. $\forall \eta \in E$. Let $x \in E$. Then:

i) $(X_{T_k})_{k \in \mathbb{Z}_{\geq 0}}$ is DTMC with transition kernel π

under p.m. P_x. starts at x.

ii) $(T_1 - T_0, T_2 - T_1, \cdots T_n - T_{n-1}, \cdots)$ are indept. when

condition on $(X_{T_k})_{k \geq 0}$. The conditional dist.

of $T_{n+1} - T_n$ is $Exp(q(X_{T_n}))$

<u>pf:</u> i) By strong Markov Property and induction:

$T_1 < T_2 < \cdots < T_n \cdots$ all finite P_x-a.s.

ii) 1°) $\eta, z \in E$. $f_1, f_2 \in B(\mathbb{R}^+)$. by strong markov at T_1:

$$E_x (I_{\{X_{T_1} = \eta, X_{T_2} = z\}} f_1(T_1) f_2(T_2 - T_1))$$

$$= E_x (I_{\{X_{T_1} = \eta\}} f_1(T_1) E_{X_{T_1}} (I_{\{T_1 = z\}} f_2(T_1)))$$

$$= \pi(x,\eta)\pi(\eta,z) \int_0^\infty e^{-q(x)s_1} f(s_1)\,ds_1 \int_0^\infty e^{-q(\eta)s_2} f(s_2)\,ds_2$$

2°) By induction:

$$E_x (\prod_1^p I\{X_{T_i} = \eta_i\} f_i(T_i - T_{i-1}))$$

$$= \pi(x,\eta_1)\pi(\eta_1,\eta_2) \cdots \pi(\eta_{p-1},\eta_p) \prod_1^p (\int_0^\infty e^{-q(\eta_{i-1})s} f_i\,ds)$$

Thm. Given $(q(x))_{x\in E}$ and $\Pi(\cdot,\cdot)$ p.m. on E. s.t. $\Pi(x,x)=0$

If $\sup_x q(x) < \infty$. $q(x) > 0$. $\forall x$. Then:

exists a corresponding Feller semigroup.

Pf: Def: $L\varphi(x) = q(x)\sum_{y\neq x}\Pi(x,y)(\varphi(y)-\varphi(x))$. $\forall \varphi \in B(E)$

Note: $\sup_x q(x) < \infty \Rightarrow L$ is BLF on $B(E)$

Directly define: $Q_t = e^{tL}$. which is Feller

Rmk: Probability Method:

Recover $(X_t)_{t\geq 0}$ from (T_n). (X_{T_n}). Set $Q_t\varphi(x) = E_x(\varphi(X_t))$

② CTMC:

From ①. Feller jump processes are CTMC

However. the universe doesn't hold!

i) Feller Process:

$$\begin{tikzcd}
 & (X_t)_{t\geq 0} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{infinitesimal generator } L \arrow[rr, leftrightarrow] & & \text{transition semigroup } (Q_t)_{t\geq 0}
\end{tikzcd}$$

ii) CTMC:

$$\begin{tikzcd}
 & (X_t)_{t\geq 0} \arrow[dl] \arrow[dr, shift left] & \\
\text{generator } P'(0) =: Q \arrow[rr, shift left, "/" marking] & & \text{transition Func. } P_{ij}(t) \arrow[ll, shift left] \arrow[ul, shift left, "/" marking]
\end{tikzcd}$$

Rmk: If S is finite. There's one-to-one correspond in ii)

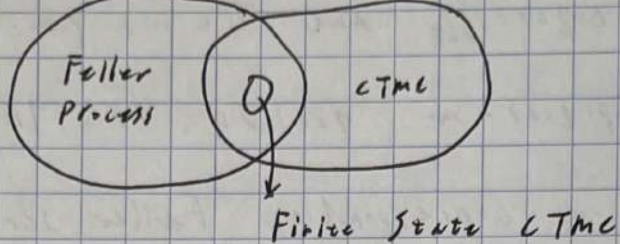
But if $|S| = S'$. there's a counterexample by Blackwell:

$S = \{0,1\}$. $\alpha,\beta \geq 0$. $\begin{cases} P_{00}(t) = \alpha/(\alpha+\beta) + \beta\exp(-(\alpha+\beta)t)/(\alpha+\beta) \\ P_{11}(t) = \beta/(\alpha+\beta) + \alpha\exp(-(\alpha+\beta)t)/(\alpha+\beta) \end{cases}$

Set $X_t = (X_t^{(1)}, \cdots X_t^{(k)}, \cdots) \in S^{\mathbb{N}}$. $(X_t^{(k)})$ indept. and $X_t^{(k)} \in S$. with parameter $\alpha_k, \beta_k \geq 0$ $\sum \alpha_k/(\alpha_k+\beta_k) < \infty$

The problem is: X_t isn't right-conti at $t=0$.

Diagram:



Rmk: Note that CTMC satisfies Strong Markov property, but it's not Feller process sometimes. Actually, Feller process has a stronger property — Feller Property (DTMC also has)

$$\text{Feller Property} \overset{i)}{\not\Leftrightarrow} \text{SMP} \overset{ii)}{\not\Leftrightarrow} \text{MP}.$$

i) $X_t = X_0$ if $X_0 \geq 0$, $X_t = X_0 - t$ otherwise.

ii) B_t is 1-dim BM, $0 \in \text{supp}(B_0)$, then consider $X_t = B_t I(B_0 \neq 0)$.

prop. $P_{xy}(t)$ is transition function for CTMC $(X_t)_{t\geq 0}$.

$P_t f(x) =: E_x(f(X_t)) = \sum_{y\in S} f(y) P_{xy}(t)$ is semigroup.

Pf: By C-K equation: $P_s P_t f(x) = P_{t+s} f(x)$.

$P_0 f(x) = f(x)$. $P_t f \in C_0$ since S equip with discrete topo.

prop. Finite States CTMC is Feller.

Pf: $|P_t f(x) - f(x)| \leq \sum_{y\in S} |f(y) - f(x)| P_{xy}(t)$

$$\lesssim 1 - P_{xx}(t) \longrightarrow 0 \quad (t \to 0)$$

Thm. (P_t) is Feller $\Leftrightarrow \lim_{x\to\infty} P_{xy}(t) = 0$, $\forall y \in S$, $t \geq 0$.

where we suppose $S = \mathbb{Z}^+$, for convention.

Pf: ($\Rightarrow$) By contradiction:

$\exists\, t_0, y_0$. s.t. $\overline{\lim_{x\to\infty}}\, p_{xy_0}(t_0) > 0$.

Set $f(x) = 2^{-x} \in C_0(S)$. $|P_{t_0} f(x)| = \sum_{y\in S} 2^{-y} p_{xy}(t) \geq \frac{p_{xy_0}(t_0)}{2^{y_0}}$

$\Rightarrow P_{t_0} f \notin C_0(S)$. Contradict!

($\Leftarrow$) For $f \in C_0(S)$. $\|f\| \leq M$. $\forall \varepsilon > 0$.

$\exists N(\varepsilon)$. $\forall y > N(\varepsilon)$. $|f(y)| < \frac{\varepsilon}{2}$.

$\exists N_0(\varepsilon)$. $\sup_{\{1,\cdots N(\varepsilon)\}} p_{xy}(t) < \varepsilon / 2MN(\varepsilon)$. $\forall x \geq N_0(\varepsilon)$.

Then: $\forall x \geq N_0(\varepsilon)$: $|P_t f(x)| \leq \sum_0^{N(\varepsilon)} + \sum_{N(\varepsilon)+1}^{\infty} \leq \varepsilon$

With: $|P_t f(x) - f(x)| \leq 2M(1 - p_{xx}(t)) \to 0\ (t \to 0)$

③ Lévy Process:

Consider $(Y_t)_{t\geq 0}$ satisfies:

i) Y_t take values in $\mathbb{R}^1$. $Y_0 = 0$ a.s.

ii) $\forall\, 0 \leq s \leq t$. $Y_t - Y_s$ indept with $\sigma(Y_r, 0 \leq r \leq s)$. $Y_t - Y_s \overset{d}{\sim} Y_{t-s}$.

iii) $Y_t \xrightarrow[P]{} 0$. as $t \downarrow 0$

e.g. i) SBM. ii) Hitting Time $(T_a)_{a\geq 0}$ of BM.

Denote: $Y_t \sim Q_t(0, dy)$. $Q_t(x, dy) = Q_t(0, dy) \circ \theta_x \sim Y_t + x$.

Prop. (Q_t) is Feller semigroup on $\mathbb{R}$. Moreover,

$(Y_t)_{t\geq 0}$ is Markov process with semigroup $(Q_t)_{t\geq 0}$.

Pf: 1°) Show: $(Q_t)_{t\geq 0}$ is transition semigroup.

Note: $(Y_{t+s} - Y_t, Y_t) \sim Q_s(0, \cdot) \otimes Q_t(0, \cdot)$

$\forall \varphi \in B(\mathbb{R})$. $Q_t Q_s \varphi(x) = \int Q_t(0, dy) \int Q_s(0, dz)\, \varphi(x+y+z)$

$$= E(\varphi(X + Y_t + (Y_{t+s} - Y_t)))$$

$$= E(\varphi(X + Y_{t+s}))$$

$$= Q_{t+s}\varphi(x).$$

Measurability of $(t, x) \mapsto Q_t(x, A)$ will follow from strong conti. of $(Q_t)_{t\geq 0}$

2°) $x \mapsto Q_t\varphi(x) = E(\varphi(x + Y_t))$ is conti by DCT.

$Q_t\varphi(x) \xrightarrow{x\to\infty} 0$. by DCT. $\Rightarrow Q_t\varphi \in C_0(E)$.

$Q_t\varphi(x) \xrightarrow{t\to 0} 0$. by ii) and iii). $\Rightarrow Q_t$ is Feller.

3°) $E(\varphi(Y_{t+s}) \mid Y_r, 0 \leq r \leq s) = E(\varphi(Y_{t+s} - Y_s + Y_s) \mid \mathcal{G}^Y_s)$

$$= \int Q_t(0, dy)\,\varphi(y + Y_s)$$

$$= Q_t\,\varphi(Y_s). \quad \forall \varphi \in B(\mathbb{R}^d)$$

$\Rightarrow (Y_t)_{t\geq 0}$ is Markov Process.

Rmk: By modification of Markov Process. $\exists (\tilde{Y}_t)$. it has càdlàg sample paths.

Def: $(Y_t)_{t\geq 0}$ take values in $\mathbb{R}^d$. is Lévy Process if it's càdlàg. satisfies i). ii).

Rmk: It satisfies iii) automatically. So it's Feller.

④ Conti-States Branching Process:

Def. A Markov $(X_t)_{t\geq 0}$ takes values in $E = \mathbb{R}^+$ with $(Q_t)_{t\geq 0}$ is called Conti-State Branching Process if $(Q_t)_{t\geq 0}$ satisfy:

$\forall x, y \in \mathbb{R}^+$. $t \geq 0$. $Q_t(x, \cdot) * Q_t(y, \cdot) = Q_t(x+y, \cdot)$.

prop. $Q_t(0,\cdot) = \delta_0(\cdot)$. i.e. zero is absorbing

Pf: Set $x = y = 0 \Rightarrow \mu * \mu = \mu$. by ch.f: $\varphi^2 = \varphi$

since $\varphi(0) = 1$. So $\varphi \neq 0 \Rightarrow \varphi \equiv 1$. i.e. $\mu = Q_t(0,\cdot) = \delta_0$.

prop. (Branching Property)

X, Y are two indept CSBP. with same semigroup $(P_t)_{t\geq 0}$ adapted to $(\mathcal{F}_t^X)$, $(\mathcal{F}_t^Y)$ resp. Then $Z = X+Y$. is also Markov Process adapted to $(\mathcal{F}_t^Z)_{t\geq 0}$. with $(P_t)_{t\geq 0}$.

Pf: Consider $\mathcal{C}_t = \{A \cap B \mid A \in \mathcal{F}_t^X, B \in \mathcal{F}_t^Y\}$. λ-class

$\forall \lambda > 0$. $A = A_1 \cap A_2 \in \mathcal{C}_s$. $A_1 \in \mathcal{F}_s^X$. $A_2 \in \mathcal{F}_s^Y$.

$$
\begin{aligned}
E(e^{-\lambda(X_{t+s}+Y_{t+s})} I_A) &= E(e^{-\lambda X_{t+s}} I_{A_1}) E(e^{-\lambda Y_{t+s}} I_{A_2}) \\
&\overset{MP.}{=} E(E_{X_s}(e^{-\lambda X_t}) E_{Y_s}(e^{-\lambda Y_t}) I_A) \\
&= E(\int e^{-\lambda(z_1+z_2)} I_A P_t(X_s, dz_1) P_t \cdots \\
&\overset{BP.}{=} E(\int e^{-\lambda z} I_A P_t(X_s+Y_s, dz))
\end{aligned}
$$

$$\Rightarrow E(e^{-\lambda Z_{t+s}} \mid \mathcal{F}_s^Z) = \int e^{-\lambda z} P_t(Z_s, dz) \in \mathcal{F}_s^Z.$$

by Monotone Class argue with $\mathcal{F}_t^X, \mathcal{F}_t^Y \subseteq \mathcal{C}_t$. $\mathcal{F}_t^Z \subseteq \mathcal{F}_t^X \vee \mathcal{F}_t^Y$

So, from inverse Laplace Transform:

Z is also Markov Process with $(P_t)_{t\geq 0}$

Rmk: Discrete State Version also has this property.

Next, we fix semigroup $(Q_t)_{t\geq 0}$. s.t.

i) $Q_t(x, \{0\}) < 1$. $\forall x > 0$. $t > 0$.

ii) $Q_t(x, \cdot) \xrightarrow{v} \delta_x(\cdot)$ when $t \to 0$.

prop. $(Q_t)_{t\geq 0}$ is Feller. Moreover. $\forall \lambda > 0 . x \geq 0$

$$\int Q_t(x, dy) e^{-\lambda y} = E_x(e^{-\lambda X_t}) = e^{-x \varphi_t(\lambda)}.$$

where $\varphi_t : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$. $\varphi_t \circ \varphi_t = \varphi_{t+s}$. $\forall s.t.$

Pf: 1) For second assertation:

$$E_x(e^{-\lambda X_t}) E_y(e^{-\lambda X_t}) = E_{x+y}(e^{-\lambda X_t}).$$

by property of BP. With: $Q_t(x.\{0\}) < 1$.

So: $E_x(e^{-\lambda X_t}) = e^{-x \varphi_t(\lambda)}$. $\varphi_t(\lambda) > 0$

2) By C-K equation:

$$\int Q_{t+s}(x.dz) e^{-\lambda z} = \int Q_t(x.dy) \int Q_s(y.dz) e^{-\lambda z}$$

$$= e^{-x \varphi_t \circ \varphi_s(\lambda)}$$

$\Rightarrow \varphi_{t+s} = \varphi_t \circ \varphi_s$.

3) Prove $(Q_t)_{t\geq 0}$ is Feller.

Let $\phi_\lambda(x) = e^{-\lambda x}$. $\Rightarrow Q_t \phi_\lambda = \phi_{\varphi_t(\lambda)} \in C_0(\mathbb{R}^+)$.

Note $(\phi_\lambda)_{\lambda \in \mathbb{R}_{>0}}$ is dense in $C(\mathbb{R}^+)$

So: $Q_t : C_0(\mathbb{R}^+) \to C_0(\mathbb{R}^+)$. BLO. $\|Q_t\| \leq 1$.

$$Q_t(\varphi(x)) = \int Q_t(x.dy)\varphi(y) \xrightarrow{t \to 0} \varphi(x).$$

by property ii) of Q_t.