Lévy Process

Fix X is a Lévy process.

<u>Rmk</u>: fBm isn't Lévy process for $H \neq 1/2$.

① Note $X_1 = \sum_k X_{k/n} - X_{k-1/n} \sim \mu_{\frac{1}{n}} * \cdots * \mu_{\frac{1}{n}}$

where μ_t is law of X_t.

$\Rightarrow$ X satisfies infinite divisibility.

<u>Prop.</u> $\mathbb{P}(\{\omega \in \Omega \mid \sup_{t \geq 0} |X_t(\omega)| < \infty\}) \in \{0,1\}$.

<u>Pf</u>: Let $g_n = \sigma(X_t - X_s,\ n \leq s \leq t \leq n+1)$

$\mathcal{G}_n = \sigma(\bigcup_{k \geq n} g_k)$

Note $A = \{\sup_{t \geq 0} |X_t| < \infty\}$

$\forall n$

$= \{\sup_{t \geq n} |X_t - X_n| < \infty\} \in \mathcal{G}_n$.

$\Rightarrow$ By Kolmogorov 0-1 law.

<u>Rmk</u>: Actually, there's no infinite divisible r.v. is a.s. bdd. unless it's const. So: $P(|X_t| \leq S < \infty) < 1$. for $\forall t \in \mathbb{R}^+$.

Pf: Otherwise. if $\mathbb{P}(|Z| \le B) = 1$.

$\forall n.\quad Z = \sum_1^n X_k.\quad X_k$, i.i.d.

$$\mathbb{P}(Z > B) \geq \mathbb{P}(X_k \geq \frac{B}{n}.\ \forall k)$$

$$= \mathbb{P}^n(X_1 \geq \frac{B}{n})$$

Symmetrically, $\mathbb{P}(X_1 < -\frac{B}{n})$

$= \mathbb{P}(X_1 > \frac{B}{n}) = 0 \Rightarrow \mathbb{P}(|X_k| \le \frac{B}{n}) = 1$

$$\mathrm{Var}(Z) = \sum_1^n \mathrm{Var}(X_k)$$

$$\le 2\sum_1^n \mathbb{E}(X_k^2) \lesssim \frac{1}{n} \to 0$$

② Thm. (Lévy-Khintchine Representation)

$\varphi_{X_t}(u) = e^{t\psi(u)}$ where $\psi(u) =$

$$iau - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}/\{0\}} (e^{iux} - 1 - iux I_{\{|x|<1\}})\, \nu(dx)$$

for $a \in \mathbb{R}^1$. $\sigma^2 \geq 0$. and ν on $\mathbb{R}^1/\{0\}$.

s.t. $\int_{\mathbb{R}^1/\{0\}} (1 \wedge |x|^2)\, \nu(dx) < \infty$. $(*)$

Rmk: i) We call (a, σ^2, ν) is a Lévy triplet for X_t.

ii) We interpret the function $\psi(u)$ by decomposing $X = X^1 + X^2 + X^3$.

three indept process. $X_t^1 = at$. $X_t^2 = \sigma B_t$

X_t^3 is jump process with jump. determind by Lévy measure ν.

iii) For d-dim: $\psi(u) = ia \cdot u - \frac{1}{2} u^T \Sigma u$

$+ \int_{\mathbb{R}^d} (e^{iu\cdot x} - 1 - \frac{iu\cdot x}{1+\|x\|}) d\nu$. $\Sigma \in \mathbb{R}^{d^2}$ positive

semidefinite. ν is Borel on $\mathbb{R}^d$. $\nu(\{0\}) = 0$

st. $\int \|x\|^2 \wedge 1 \, d\nu(x) < \infty$.

Conversely, if such (Σ, b, ν) ~~exists~~.

Then. $\exists$ corresp. Lévy process.

iv) Apply Taylor expansion on $\int \frac{\square}{A}$.

$(*)$ is just for the integrability.

v) Recall the charac. of infinite divisible r.v. and their ch.f's.

In fact. $\forall$ i.d. r.v. Y. $\exists$ Lévy process X_t. st. $X_1 = Y$.

iv) Drop and stationary incre. We also have inhomo. result:

$\mathbb{E}(e^{i\lambda\cdot(X_t-X_0)}) = e^{i\psi_t(\lambda)}$. where $\psi_t(\lambda)$

$= i\lambda\cdot b_t - \frac{1}{2}\lambda^T\Sigma_t\lambda + \int_{\mathbb{R}^n\times[0,t]}(e^{i\lambda\cdot x}-1-i\lambda\cdot x/(1+\|x\|))\,d\mu(x,s)$. Σ_t, b_t, μ unique

Σ_t, b_t. conti. $\Sigma_0 = b_0 = 0$. $\Sigma_t - \Sigma_s \geq 0$.

$\int_{\mathbb{R}^n\times[0,t]}\|x\|^2\wedge 1\,d\mu < \infty$. $\mu(\{0,t\}\times\mathbb{R}^n) = 0$.

Conversely, the ch.f also corresp. □

Thm[1]. ∀ Lévy process has càdlàg modification

Thm[2]. ∀ càdlàg Lévy process is a semimart.

Pf: By LK decompose. We see it

$= at + \sigma B_t + Y$.

And Jump process Y is semimart.

Rmk: Thm[1] & Thm[2] also holds when dropping stat. incre. cond. and X_t also has decompose: $b_t + W_t + Y_t$ as in Rmk iv)

Rmk: We can define pre-Poisson process N_t from Lévy process. s.t. $N_t - N_s \sim \mathrm{Poi}(\lambda(t-s))$

And modify it as the case of BM

But note that its value space is $\mathbb{Z}^1$.

So we define:

Poisson process is pre-Poisson process with càdlàg sample path.

Lemma. Càdlàg path $f(t)$ has at most countable infinite jump.

Pf: Prove: $\forall [n, n+1)$. f has at most finite jumps. s.t. $|\Delta f| \geq \varepsilon$

Otherwise. $\exists (s_{n_k})$. subseq of jumps. s.t. $s_{n_k} \uparrow s$.

Set $t_k \in (s_{n_k}, s_{n_{k+1}})$. $|s_{n_{k+1}} - t_k|$ small enough. s.t. $|f(t_k) - f(s_{n_{k+1}}-)| < \frac{\varepsilon}{2}$. $\Rightarrow |f(t_k) - f(s_{n_{k+1}})| \geq \frac{\varepsilon}{2}$.

Set $k \to \infty$. contradiction!