Stochastic Diff. Equations

(1) Def & Examples:

For $b : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$

Consider $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$, where B_t is m-dim BM.

Def: For b, σ locally bdd, measurable.

i) $((\Omega, \mathcal{F}, \mathbb{P}), \xi, X, B)$ is called weak Sol. for SDE above if:

a) $(\Omega, \mathcal{F}, \mathbb{P})$ is filted prob. space st. $\xi \in \mathcal{F}_0$. X_t & B_t are $\mathcal{F}_t$-adapted.

b) $X_t^i = \xi + \int_0^t b_i(t, X_s)ds + \int_0^t \sum_j \sigma_{ij}(\cdots)dB_t^j$

i.e. integral solution.

Rmk: i) The Sto-integral in b) is well-def since b, σ local bdd.

ii) $X_0 = \xi$ is indept of $(B_t)_{t\geq 0}$ by Blumenthal 0-1 law. So it can be prescribed.

ii) X is strong sol. given B_t if X is weak sol. s.t. $\mathcal{F}_t = \sigma(B_s, s \le t, \xi)$

iii) Pairwise (strong) uniqueness holds for SDE if $\forall X, Y$ solves SDE with ξ

$\Rightarrow \mathbb{P}(\forall t \geq 0,\ X_t = Y_t) = 1$.

iv) Weak uniqueness holds for SDE if $\forall$ weak sol. $((\Omega^i, \mathcal{F}^i, \mathbb{P}^i), \xi^i, B^i, X^i)$ s.t. $\mathcal{L}_{\xi^1} = \mathcal{L}_{\xi^2}$ have same finite-dim distribution (i.e. has same law)

<u>Rmk</u>: It implies: $E^1(\varphi(X^1)) = E^2(\varphi(X^2))$, $\forall \varphi \in C_B(C([0,T], \mathbb{R}^n), \mathbb{R})$

<u>e.g.</u>, (Tanaka equation)

Consider SDE: $dX_t = \mathrm{sgn}(X_t)\, dB_t$, $X_0 = 0$.

i) Weak unique hold: $\langle X \rangle_t = t$. So: it has law as BM.

ii) Weak exist hold: Set X_t is BM.

$\Rightarrow B_t = \int_0^t \mathrm{sgn}(X_t) X_t$ is also BM.

Note $X_t = \int_0^t |sgn(X_t)|^2 dX_t$

$= \int_0^t sgn(X_t) dB_t$.

$\Rightarrow ((\Omega), X_t, B_t)$ is weak sol.

iii) Strong unique fails: if X solves the SDE, then $-X$ also solves it. since $\int_0^\infty I_{\{X_s=0\}} ds = 0$ a.s.

iv) Strong exist fails: By contradict, from Itô-Tanaka formula:

$B_t = \int_0^t sgn(X_t) dX_t = |X_t| - L_t^0(X)$

Note $L_t^0(X) = |X_t| - \lim_{h\to 0} Leb\{s \in [0,t], |X_s| \le h\} / h$

$\Rightarrow B_t \in \mathcal{F}_t^{|X|}$. So: $\mathcal{F}^B \subset \mathcal{F}^{|X|}$

So: $\mathcal{F}^X \subset \mathcal{F}^{|X|}$. contradiction!

(e.g. $sgn(X_t) \in \mathcal{F}^X$, but $\notin \mathcal{F}^{|X|}$, otherwise $sgn(X_t) = F((|X_s|)_{s\le t}) \Rightarrow$ $sgn(X_t) = -sgn(X_t)$. contradict!)

Rmk: iii) fails because $\mathcal{F}^X$ is larger than $\mathcal{F}^B$ reconstruction from B doesn't work.

(2) Strong Exist & Uniqu:

Thm. (Pairwise uniqu)

If b, σ are locally bdd and locally mono. i.e. $2(b(t,x)-b(t,y))\cdot(x-y)+|\sigma(t,x)-\sigma(t,y)|^2 \le K_{T,n}|x-y|^2$. $\forall t\in[0,T]$, $|x|,|y|\le n$

where $|\cdot|$ is Frobenius norm $(|x|^2=\sum_{i,j}x_{ij}^2)$

Then pairwise uniqu holds for SDE.

Pf: Set $\tau_n=\inf\{t\ge 0 \mid |X_t|\vee|Y_t|\ge n\}$

Apply Itô's on $Z_t=|X_t^{\tau_n}-Y_t^{\tau_n}|^2$.

$\mathbb{E}(Z_t)=2\mathbb{E}(\int_0^{t\wedge\tau_n}\Box)$

$\overset{\text{cond.}}{\le}\mathbb{E}(\int_0^t K_{T,n}Z_s\,ds)$

By Gronwall's $\Rightarrow Z_t=0$. a.s. $\forall t$.

Note X_t, Y_t. conti. $\tau_n\uparrow\infty$. So:

$\mathbb{P}(X_t=Y_t.\ \forall t\ge 0)=1$.

Pf: For $\lambda\ge 0$. Set $E_\lambda := \{(H_t)$ conti. $\mathcal{F}^{B.s}$-adapted. $\mathbb{R}^d$-valued. $\|H\|_{E_\lambda}<\infty\}$ where

$$\|H\|_{E_\lambda} := \sup_{t\geq 0} e^{-\lambda t} E(\sup_{[0,t]} |H_s|^2)^{\frac{1}{2}}.$$

Rmk: E_λ is Banach space. (CLS of L^2)

Thm. (Strong well-posedness)

If σ, b satisfy globally Lipschitz:

$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq k|x-y|$

$\forall x, y \in \mathbb{R}^d$, $t\geq 0$. and globally coercive:

$|b(t,x)| + |\sigma(t,x)| \leq k(1+|x|)$. $\forall x\in\mathbb{R}^d$, $t\geq 0$

Then $\forall \mathcal{F}_0^B$-measurable initial $\xi < \infty$ a.s.

$\exists$ unique strong sol. to the SDE.

Rmk: Coercive $\not\Rightarrow$ Lip. (the const got from triangle inequ. depend on t)

Pf: 1) Assume $\xi \in L^2(\Omega)$.

$\mathcal{I}(H) := \xi + \int_0^\cdot b(s, H_s) ds + \int_0^\cdot \sigma(\cdot\cdot) dB_s$

Note $\mathcal{I}: E^\lambda \to E^\lambda$ conti. by coercive.

Next, we use Banach fixed pt Thm to prove $\exists X \in E_\lambda$, s.t. $\mathcal{I}(X) = X$.

$$\| \sup_{[0,t]} | \mathcal{Z}(X^1)_r - \mathcal{Z}(X^2)_r | \|_{L^2} \overset{\text{Minkow}}{\leq} I_t^1 + I_t^2.$$

$$I_t^1 := \int_0^t \| b(s, X_s^1) - b(s, X_s^2) \|_{L^2} \, ds$$

$$\overset{\text{Lip}}{\leq} k \int_0^t \| X_s^1 - X_s^2 \|_{L^2} \leq k \| X^1 - X^2 \|_{E^\lambda} \int_0^t e^{\lambda s}$$

$$= k e^{\lambda t} \| X^1 - X^2 \|_{E^\lambda} / \lambda.$$

$$I_t^2 := \| \sup_{[0,t]} | \int_0^t | \sigma(s, X_s^1) - \sigma(s, X_s^2) | \, dB_s \|_{L^2}$$

$$\overset{\text{BDG}}{\leq} C \, \mathbb{E} \left(\int_0^t | \sigma(\cdot\cdot) - \sigma(\cdot\cdot) |^2 dt \right)^{1/2}$$

$$\overset{\text{Lip}}{\leq} C k \| X^1 - X^2 \|_{E^\lambda} \left(\int_0^t e^{2\lambda s} ds \right)^{\frac{1}{2}}.$$

$$\Rightarrow \| \mathcal{Z}(X^1) - \mathcal{Z}(X^2) \|_{E^\lambda} \leq \left(\frac{k}{\lambda} + \frac{ck}{\sqrt{2\lambda}} \right) \| X^1 - X^2 \|_{E^\lambda}$$

Choose λ large enough. $\Rightarrow \mathcal{Z}$ is contraction

2) For $\xi \in \mathcal{F}^0$:

Note $\exists X^n$ unique strong solution for

$$dX_t^n = b(t, X_t^n) dt + \sigma(t, X_t^n) dB_t.$$

$$X_0^n = \xi \, I\{|\xi| \leq n\}.$$

And note $X_t^m I\{|\xi| \leq n\}$ also solves it.

for $\forall m \geq n$. $\Rightarrow X_t^m = X_t^n$ on $\{|\xi| \leq n\}$.

Set $X_t = X_t^n$ on $\{|\xi| \leq n\}$. $\mathbb{P}(\bigcup_n \{|\xi| \leq n\}) = 1$.

e.g. (Ornstein-Uhlenbeck process)

$d = m = 1$. $a, \sigma > 0$, $\mu \in \mathbb{R}$: all const. $x \in \mathbb{R}$

Consider $dX_t = a(\mu - X_t)dt + \sigma dB_t$. $X_0 = x$.

Rmk: It has mean-reversion behavior.

To solve it: split its homogeneous part

$dY_t = -a Y_t dt$. $\Rightarrow Y_t = c e^{-at}$.

Let $Z_t = X_t / Y_t$. Then Z_t solves:

$dZ_t = \mu a e^{at} dt + \sigma e^{at} dB_t$.

So: $X_t = e^{-at} x + \mu(1 - e^{-at}) + \int_0^t \sigma e^{-a(t-s)} dB_s$

after obtaining Z_t.

Rmk: X_t is Gaussian process.

e.g. (ODE in BM-time).

For $\varphi(t)$ solve ODE: $d\varphi/dt = f(\varphi(t))$

Shift by B_t: $\varphi(B_t)$ solves $d\varphi(B_t)$

$= \varphi'(B_t) \circ dB_t = f(\varphi(B_t)) \circ dB_t$

$= f(\varphi(B_t)) dB_t + \frac{1}{2}(\nabla f f)(\varphi(B_t)) dt$.

Rmk: If $f \in C^2$. Lip $\Rightarrow X_t = \varphi(B_t)$ is its unique sol.

e.g. $X_t = \varphi(B_t)$. $\varphi = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ BM on unit circle. $\Rightarrow f = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

Thm. (path-depend SDE).

Consider $b, \sigma : \mathbb{R}^{\geq 0} \times C(\mathbb{R}^{\geq 0} ; \mathbb{R}^d) \to \mathbb{R}^d, \mathbb{R}^{d \times m}$

If $\exists c > 0$. s.t. $\forall t \geq 0$. $x, y \in C(\mathbb{R}^{\geq 0} ; \mathbb{R}^d)$.

$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K \|x-y\|_{\infty, [0,t]}$

& $|b(t,x)| + |\sigma(t,x)| \leq K(1 + \|x\|_{\infty,[0,t]})$

Then strong exist & unique hold for

$dX_t = b(t, (X_t)_{t \geq 0}) dt + \sigma(t, (X_t)_t) dB_t$. $X_0 = x$

Rmk: The sol. is no longer Markovian!

(3) Weak exist & unique:

Thm. (Skorokhod)

If $b(t,x)$. $\sigma(t,x)$ conti. bdd. Then:

$\exists$ weak sol. for the SDE

Rmk: For $b = b(x)$. $\sigma = \sigma(x)$. We can only require b, σ are measurable & bdd.

Thm (Stroock - Varadhan)

For $d = m$. If b is measurable bdd &

σ is conti. bdd. s.t. $\forall t \geq 0$. $x \in \mathbb{R}^d$. $\exists \varepsilon(t,x) > 0$. s.t. $|\sigma(t,x) v| \geq \varepsilon(t,x)|v|$. $\forall v \in \mathbb{R}^d$.

Then weak exist & weak unique hold.

e.g. (Regularization by noise)

Note for SDE $dX_t = b(t, X_t)dt + dB_t$.

$X_0 = x \in \mathbb{R}^d$. if $\mathbb{E}(e^{\frac{1}{2}\int_0^T |b(s,X_s)|^2 ds}) < \infty$

(Novikov cond.) holds. then its weak exist & unique in law hold.

Pf: 1) Let $X_t = W_t + x$. W_t is $\mathbb{P}$-Bm.

Set $dQ/dP = \mathcal{E}(\int_0^\cdot b(s,X_s) dW_s)$

$\Rightarrow X_t = x + \int_0^t b(s,X_s) ds + \widetilde{W}_t$.

$\widetilde{W}_t$ is Q-Bm. $\Rightarrow (Q, X_t, \widetilde{W}_t)$

is weak sol. for the SDE.

ii) Set $d\tilde{Q}/dP = \mathcal{E}(-\int_0^{\cdot} b(s, X_s) dB_s)_\infty$

$\Rightarrow$ weak sol. $X_t = X_0 + B_t + \int_0^t b(\cdots) ds$

$= X_0 + \tilde{W}_t$. $\tilde{W}_t$ is $\tilde{Q}$ - BM

With CM formula, its dist. unique.

<u>Cor.</u> Under cond. above. $dX_t = b(X_t)dt + \operatorname{sgn}(X_t) dB_t$ also has weak sol. and unique in law.

<u>Rmk</u>: i) If $|\sigma| + |\sigma^{-1}| \le K$. $\forall t, x$. In addition

$$dX_t = b(s, X_s) ds + \sigma(s, X_s) dB_s$$

$$= \sigma(s, X_s)(\sigma^{-1} b(s, X_s) ds + dB_s)$$

We see how Stroock-Varadhan works.

ii) If b is bdd drift, then path-wise uniqueness also holds.

Let $b(x) = I_{[0]}(x)$. $x_0 = 0$. We see that $dX_t = I_{[0]}(X_t) dt$. $X_0 = 0$ has no solution.

But $dX_t = I_{[0]}(X_t) dt + dB_t$ has sol.

This is kind of regularization by noise.

(4) Yamada Theory:

Thm. (Yamada-Watanabe & Cherny)

For $b, \sigma(t,x)$ locally bdd measurable.

i) Weak exist & strong unique $\Rightarrow$ strong exist.

ii) Strong exist & weak unique $\Rightarrow$ strong unique.

Thm. (Yamada-Watanabe)

For $d=1$. If $\exists\, \gamma > 0$, $k, h: \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ $\uparrow$

s.t. $\int_0^{\gamma} ds/h^2(s) = \infty$. $\int_0^{\gamma} ds/k(s) = \infty$. k concave

$|b(t,x)-b(t,y)| \leq C|x-y|$. $\forall t \geq 0$, $\forall x,y \in \mathbb{R}^1$.

$|\sigma(t,x)-\sigma(t,y)| \leq h(|x-y|)$. $\forall t \geq 0$. $x,y \in \mathbb{R}^1$.

Then: pathwise uniqueness holds. $\forall \xi \in \mathcal{F}_0$

Rmk: When $d \geq 2$. h has to satisfy $\int_0^{\gamma} u/h^2(u)\, du = +\infty$. and h is sub-additive (which's nearly Lip.).

e.g. for $h(u) = |u|^\alpha$. $\alpha < 1$. pathwise uniqueness doesn't hold.

prop. (Comparison principle)

For $d=1$. b_i. σ are Lip. s.t. $b_1(t,x) \geq b_2(t,x)$ $\forall t,x$. & $\xi_1 \geq \xi_2$. a.s. $\xi_i \in \mathcal{F}_0$. $<\infty$. a.s. $\Rightarrow$ for X^i unique strong sol. to $dX_t^i = b_i(t, X_t^i)dt + \sigma(t, X_t^i)dB_t$. $X_0^i = \xi^i$. we have: $P(X_t^1 \geq X_t^2.\ \forall t \geq 0) = 1$.

Rmk: It only works in $d=1$. since it depend on the order struc. of $\mathbb{R}'$.

Pf: Let $Y_t = X_t^2 - X_t^1$. $f(y) = (y^+)^3$. $\tau_n :=$ $\inf\{t \geq 0 \mid |X_t^1| \vee |X_t^2| \geq n\}$. Apply Itô on $f(Y_t^{\tau_n})$

$$\Rightarrow E(f(Y_t^{\tau_n})) \overset{b_1 \geq b_2}{\leq} 3\int_0^t E(\square)$$

$$\overset{Lip}{\leq} C_t \int_0^t E(f(Y_t^{\tau_n}))$$

Apply Gronwall's inequi. $f(Y_t^{\tau_n}) = 0$. a.s.

So. $\forall t \in \mathbb{Q}$. $Y_t^{\tau_n} = 0$. a.s. Let $n \uparrow \infty$.

with X. Y conti. $P(Y_t = 0.\ \forall t \geq 0) = 1$.

(5) <u>blowup & local sol.</u> :

Note that when b, σ are only locally Lip. $\Rightarrow$

its sol. can explode in finite time.

<u>Def</u>: b, σ: measurable locally bdd.

i) $((\Omega, \mathcal{F}, P), y, B, X, \tau)$ is local weak sol. to the SDE on $[0, \tau]$ if

a) $(\Omega, \mathcal{F}, P)$ is filtered prob. space. & τ is $(\mathcal{F}_t)$-stopping time. X_t is $(\mathcal{F}_t)$-adapted.

b) $X^i_{t\wedge\tau} = y + \int_0^{t\wedge\tau} b_i(s, X_s)ds + \int_0^{t\wedge\tau} \sum_j \square dB^j_s$

ii) X_t is local strong sol. if it's a weak sol. & $\mathcal{F}_t = \mathcal{F}^{B, y}_t$.

iii) Strong local unique for (X^i, τ^i) is $X^1 = X^2$ on $[0, \tau^1 \wedge \tau^2]$.

iv) Weak local unique : $X^1 \sim X^2$ on $[0, \tau^1 \wedge \tau^2]$.

<u>Thm</u>. If b, σ are locally bdd & locally Lip:

$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le k_{T,n} |x-y|$

for $\forall t \in [0, T]$. $|x|, |y| \le n$. Then, $\forall \xi < \infty$ a.s. $\in \mathcal{F}_0$, $\exists!$ local strong sol. to the SDE till $\tau_U := \inf\{t \ge 0 \mid X_t \in U^c\}$. for $\forall U$ open bdd in $\mathbb{R}^d$.

<u>Pf</u>: Choose cutoff func. $\varphi_n(t, x) = 1$ on $[0, n] \times \{|x| \le n\}$. $\varphi_n \equiv 0$ $\forall t \ge n+1$ or $|x| \ge n+1$

Set $b_n = b\varphi_n$, $\sigma_n = \sigma\varphi_n$

$\Rightarrow (b_n, \sigma_n)$ is globally bdd & Lip.

$\Rightarrow \exists!$ strong unique sol. X^n.

Note $b_n = b_m$, $\sigma_n = \sigma_m$. $\forall t \le n$, $|x| \le n$ for $\forall m \ge n$. $\Rightarrow X^n = X^m$ on $\square$.

Set $X_{t \wedge \tau_{B_n}} = X^n_{t \wedge \tau_{B_n}}$ is local strong sol. on $[0, \tau_{B_n}]$.

And $\forall U$ bdd open. $\exists B_n$. s.t. $U \subset B_n$

So restrict X^n on $[0, \tau_U]$.

<u>Rmk</u>: $b, \sigma \in C^1 \Rightarrow b, \sigma$ locally Lipschitz.

<u>Def</u>: $\tau^* := \lim_{n \to \infty} \tau_{B_n}$ (above) is called explosion

time. And (X, ζ^*) is called the maximal sol. to the SDE.

Rmk: $\sup_{[0,\zeta^*]} |X_t| < \infty$. $\lim_{t\to\zeta^*} |X_t| = \infty$. So

X_t can't live outside $[0, \zeta^*)$.

If $\mathbb{P}(\zeta^* = \infty) = 1$. We called (X, ζ^*) is global solution.

Lemma. $\mathcal{L}$ is generator of the SDE. If $\exists$ $V \in C^2(\mathbb{R}^d; \mathbb{R}^{\geq 0})$, $k_T > 0$. St. $\mathcal{L}V \leq k_T(1+V)$ $\forall t \in [0, T]$. $x \in \mathbb{R}^d$.

Then: $\mathbb{E}(V(X_t^\zeta)) \leq (k_t t + \mathbb{E}(V(\xi))) e^{k_t t}$.

where (X, ζ) is local sol. $V(\xi) \in L^1 \cap \mathcal{F}_0$.

Pf. Apply Itô's on $V(X_t^{\tilde{\zeta}_n})$. and then use Gronwall's inequi. let $n \uparrow \infty$.

where $\tilde{\zeta}_n = \zeta \wedge \inf\{t \geq 0 \mid |X_t| \geq n\}$.

Rmk: V is kind of Lyapunov func. if $V(x) = |x|^2$. then the cond is called weakly coercive, i.e.

$$2b(t,x)\cdot x + |\sigma(t,x)|^2 \le K_T (1+|x|^2).$$

Thm. If the cond. above holds for some V and b, σ satisfy locally Lip. cond. Then. $\forall \xi \in \mathcal{F}_0$. $\mathbb{P}(\tau^* = \infty) = 1$.

Pf: Set local time or ξ s.t. $V(\xi) \in L^1$

$$\mathbb{E}(V(X_{m\wedge t})) \overset{\text{Fatou}}{\le} \underline{\lim}\, \mathbb{E}(V(X_{m\wedge\tau_n}))$$

$$\le (\mathbb{E}(V(\xi)) + K_m m) e^{K_m m} < \infty.$$

So $\tau^* \overset{a.s.}{>} m$. $\forall m$. Since $X_{\tau^*} = \infty$.

$$\Rightarrow \mathbb{P}(\tau^* = \infty) = \lim_{m\to\infty} \mathbb{P}(\tau^* > m) = 1.$$

Then remove local time as before.

e.g. (Logistic growth model)

$$dX_t = r(1-X_t)X_t\,dt + \mu X_t\,dB_t.\quad X_0 = x.$$

Note the coeff are both locally Lip.

$\Rightarrow$ pathwise uniqueness holds.

if $x=0$. then $Y \equiv 0$ the one solution.

Apply comparison prin. $X_t \ge 0$. when $x \ge 0$

Then since $b(x)x \le rx^2$. $\underset{b=rx}{\overset{\text{if } x>0}{\Rightarrow}}$ sol. won't blowup.