① We want to describe the ODE below:

$$\frac{d}{dt} X_t = b(X_t) + \xi_t, \quad X_0 = x_0.$$

<u>Lemma.</u> If $(\xi_t)_{t \in \mathbb{R}^{\geq 0}}$ is centered Gaussian process with cov. $\Gamma(s,t) = I_{\{s=t\}}$

Then: $\Omega \times [0,t) \ni (w,s) \mapsto \xi_s(w) \in \mathbb{R}^1$ isn't measurable w.r.t. $\mathcal{F} \otimes \mathcal{B}_{[0,t)}$.

And $\int_0^t \xi_s(w)\,ds$ might not be def!

<u>Pf:</u> By contradiction, we have:

$\Omega \times [0,t)^2 \ni (w, s_1, s_2) \mapsto \xi_{s_1}(w)\xi_{s_2}(w)$ is measurable w.r.t $\mathcal{F} \otimes \mathcal{B}^{\otimes 2}_{[0,t)}$.

Note $\int_0^r \int_0^r \mathbb{E}(|\xi_{s_1}\xi_{s_2}|)\,ds_1 ds_2 \leq r^2$.

So with measurable. We apply Fubini:

$$\mathbb{E}\left(\left(\int_0^r \xi_s\,ds\right)^2\right) = \int_0^r \int_0^r I_{\{s_1 = s_2\}}\,ds_1 ds_2 = 0$$

$$\Rightarrow \int_0^r \xi_s\,ds = 0, \ \forall r > 0 \overset{\text{Approx.}}{\Rightarrow} \xi_s \overset{a.s.}{=} 0$$

But $\mathbb{E}\left(\int_0^t |\xi_s|\,ds\right) = \int_0^t \mathbb{E}(|\xi_s|)\,ds = \sqrt{\frac{2}{\pi}}\,t < \infty.$

To avoid this problem, we assume $(\xi_t)_{t\geq 0}$ is i.i.d. $N(0,\infty)$ rather than $N(0,1)$.

by letting $(\xi_t)_{t\geq 0}$ is centered Gaussian process with $\Gamma(s,t) = \delta(t-s)$.

where $\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$ Dirac delta.

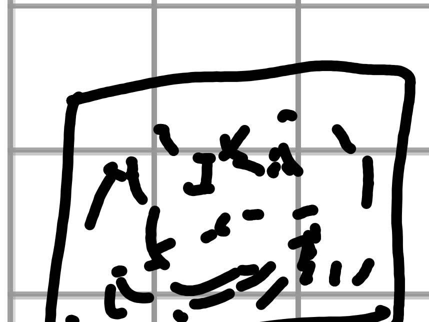
And set $\xi(f) = \int_0^\infty \xi_t f(t)\,dt$ on $L^2(\mathbb{R}^+)$.

$\Rightarrow E(\xi(f)\xi(g)) \overset{\text{Fubini}}{=} \int_{\mathbb{R}^+} f g \, dt = \langle f, g\rangle_{L^2}$.

Let $(\xi(f))_{f\in L^2(\mathbb{R}^+)}$ with cov. $\Gamma(f,g) := \langle f, g\rangle_{L^2(\mathbb{R}^+)}$ called white noise.

And $X_t = x_0 + \int_0^t b(X_s)\,ds + \overbrace{\int_0^t \xi_s\,ds}^{= \xi(I_{[0,t]})} \sim B_t$.

Remark: i) Consider (e_n) o.n.b. of $L^2(\mathbb{R}^+)$ and $(X_n) \overset{\text{i.i.d.}}{\sim} N(0,1)$. We can construct white noise $\xi(f) := \sum_1^\infty \langle f, e_n\rangle_{L^2} X_n$



television.

It means that white noise ξ equally contribute to all frequencies.

Which explains its name. (like white light)

ii) Set $B_t^{(n)} = \sum I_{[\frac{k}{2^n}, \frac{k+1}{2^n})} (B_{\frac{k}{2^n}} + 2^n (t - \frac{k}{2^n}) B_{\frac{k}{2^n}, \frac{k+1}{2^n}})$

$$\int_0^\infty f(t)\, dB_t = \lim_{n\to\infty} \int_0^\infty f(t) \frac{\partial B_t^{(n)}}{\partial t}\, dt$$

$$= \lim_{n\to\infty} \int_0^\infty f(t)\, \xi_t^{(n)}\, dt.$$

We can also interpret ξ as limit of $\xi_t^{(n)} = \sum I_{[\frac{k}{2^n}, \frac{k+1}{2^n})} 2^n B_{\frac{k}{2^n}, \frac{k+1}{2^n}}$.

Note $\mathbb{E}(\xi_t^{(n)}) = 0$. $Var(\xi_t^{(n)}) = 2^n \to \infty$

So ξ_t is also i.i.d $N(0, \infty)$.

iii) The 4^{th} method to get (ξ) is Wiener integral. ($\int f\, dB$, $f \in L^2(\mathbb{R}^{\geq 0})$. from approxi.)

③ Note that because of bad path property of pre-BM $B_t = \int_0^t d\xi_s$. $t \mapsto B_t(\omega)$ may not be measurable.

So, then we introduce definition "modification" and Kolmogorov Lemma to construct BM. with conti. path. $\forall \omega \in \Omega$.

Rmk: i) $\mathcal{G}_t$-adapted & $\mathbb{F} \times \mathcal{B}_{\mathbb{R}^+}$-measurable $\Rightarrow$ It has progressive measurable modificate

ii) Modification of conti. process may

not be still conti.

iii) Continuity can't be criticized by finite-dim law of process: e.g.

For conti. X_t. Set $Z \sim U[0,1]$ &

$\widetilde{X_t}(w) = X_t(w) + I_{\{Z(w)\}}(t)$

$\Rightarrow \widetilde{X_t}$ has discontinuity. $\forall w \in \Omega$

But $\mathbb{P}(X_t = \widetilde{X_t}) = 1$. $\forall t$. (So finite-dim)

And even $C(\mathbb{R}^+; \mathbb{R}^1)$ or $\{f : t \mapsto 0\}$

both not measurable w.r.t. $\mathcal{B}_{\mathbb{R}^1}^{\otimes \mathbb{R}^1}$

$1 = \mathbb{P}(X \in C(\mathbb{R}^+; \mathbb{R}^1))$

$= \mathbb{P}(\widetilde{X} \in C(\mathbb{R}^+; \mathbb{R}^1)) = 0$. Contradict!

Remark: i) The problem is law of B_t def on $\mathcal{B}(\mathbb{R})^{\mathbb{R}^+}$: $A \in \mathcal{B}(\mathbb{R})^{\mathbb{R}^+} \Leftrightarrow$

for countably many t_k, we have:

$A = \{(W_{t_1}, \cdots W_{t_k} \cdots) \in B\}$. for some $B \in \mathcal{B}_{\mathbb{R}}^{\otimes \mathbb{N}}$

But we need to check $\forall t \in \mathbb{R}^+$

ii) Or we can consider $\mathcal{B}$ on $(C(\mathbb{R}^+; \mathbb{R}^1), \mathcal{B}(C(\mathbb{R}^+; \mathbb{R}^1)))$ rather than $(\mathbb{R}^{\mathbb{R}^+}, \mathcal{B}(\mathbb{R}^1)^{\mathbb{R}^+})$. Via Donsker's

Thm. (law of iterated log)

i) A.s. $\forall T \in \mathbb{R}^+$. $\lim_{r\to 0} \sup_{\substack{t\in[0,T] \\ |s-t|\le r}} \frac{|B_t - B_s|}{\sqrt{2r\log 1/r}} = 1$.

ii) $\overline{\lim_{t\to 0}} \frac{B_{r+t} - B_r}{\sqrt{2t\log\log 1/t}} = 1$. $\underline{\lim} A = -1$. a.s.

⑧ Actually mart. property of BM is determined by choice of filtration $(\mathcal{F}_t)$.

If W_t is $\mathcal{F}_t$-mart. BM on $(\Omega, (\mathcal{F}_t), \mathbb{P})$.

i) If $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(W_T)$.

$\Rightarrow$ W_t is a conti. semimart on $(\Omega, (\mathcal{G}_t), \mathbb{P})$. which's called Brownian bridge on $[0,T]$. And it has decomposition:

$$W_t = \widetilde{W}_t + \int_0^{t\wedge T} \frac{W_T - W_s}{T-s} ds. \quad \widetilde{W}_t \sim (\Omega, \mathcal{G}_t, \mathbb{P})\text{-BM}.$$

Pf: 1) Note $\mathbb{E}(W_t - W_s | \mathcal{G}_s) = \frac{t-s}{T-s}(W_T - W_s)$.

2) $\widetilde{W}_t := W_t - \int_0^{t\wedge T} \frac{W_T - W_s}{T-s} \Rightarrow \langle \widetilde{W} \rangle_t = t$

3) $\mathbb{E}(\widetilde{W}_t - \widetilde{W}_s | \mathcal{G}_s)$

$= \mathbb{E}(W_{s,t} - \int_s^t (W_T - W_u)/(T-u)\, du | \mathcal{G}_s)$

$$\overset{\text{Fubini}}{=} \frac{t-s}{T-s}(W_T - W_s) - \int_s^t \mathbb{E}\left(\frac{W_T - W_u}{T-u} \Big| \mathcal{G}_s\right) du$$

$$\overset{\text{i)}}{=} \square - \int_s^t \left(\frac{T-s}{T-s} W_{s,T} - \frac{u-s}{T-s} W_{s,T}\right) \Big/ (T-u) \, du$$

$= 0$. $\Rightarrow$ By Lévy charac. $\widetilde{W}_t$ is BM.

iii) If $\mathcal{H}_t := \mathcal{F}_t \vee \sigma(W_s, s \le T)$.

$\Rightarrow$ W_t isn't even a conti. semimart on

$(\Omega, \mathcal{H}_t, \mathbb{P})$ constricted on $[0, T)$.

Pf: Note if W is semimart. $W = M + A$.

$\mathbb{E}(W_t | \mathcal{H}_0) = \mathbb{E}(W_t | \sigma(W_s, s \le T)) = W_t$. $\forall t \le T$.

So: $W_t = \mathbb{E}(A_t | \mathcal{H}_0)$. Set $(t_k^n) \overset{\Delta}{=} (kt/2^n)$.

$t = \langle W \rangle_t = \lim_{n\to\infty} \square$ (along t_k^n) $= \langle \mathbb{E}(A_\cdot | \mathcal{H}_0) \rangle_t$

$\le 2(\langle \mathbb{E}(A_\cdot^+ | \mathcal{H}_0) \rangle_t + \langle \mathbb{E}(A_\cdot^- | \mathcal{H}_0) \rangle_t)$

$$\langle \mathbb{E}(A_\cdot^\pm | \mathcal{H}_0) \rangle_t \overset{\text{Jensen}}{\le} \lim_n \sum^{2^n} \mathbb{E}(|A^\pm_{t_k^n} - A^\pm_{t_{k-1}^n}|^2 | \mathcal{H}_0)$$

$$\overset{\text{mono}}{=} \mathbb{E}(\langle A_\cdot^\pm \rangle_t | \mathcal{H}_0) = 0. \text{ Contradiction!}$$

Rmk: Semimart. is more stable under the change of equi. p.m.

④ fBM isn't semimart. if $H \neq 1/2$.

Thm. (Bichteler Dellacherie)

For càdlàg adapted process S_t:

i) S_t is conti. semimart.

ii) $\{\int_0^t \xi \, dX \mid \xi$ is elementary, $|\xi| \leq 1\}_{t>0}$ is bdd in pr.

iii) $\forall \, m_t^n = \sum_1^{N_n} \tau_k^n I_{(t_{k-1}^n, t_k^n]}$, $\tau_k^n \in \mathcal{F}_{t_{k-1}^n}$ simple pred. process s.t. $m_t^n \xrightarrow{ucp} 0 \Rightarrow$ $\int_0^t m \, ds \xrightarrow{pr} 0$.

We have i), ii), iii) equi.

Return to the proof:

Set $\tau_1^n = 0$, $\tau_k^n = n^{2H-1} (B_{t_{k-1}^n}^H - B_{t_{k-2}^n}^H)$, where $t_k^n = kT/n$. By $(H-\varepsilon)$-Hölder $\Rightarrow m^n \xrightarrow{ucp} 0$.

since B^H is H-stable, i.e. $B_{\lambda t}^H \overset{d}{\sim} \lambda^H B_t^H$ for $\forall \lambda > 0$. (Check by ch.f.)

We have: $\sum_1^n n^{2H-1} (B_{..}^H - B_{...}^H) (B_{..}^H - B_{..}^H) \overset{d}{\sim}$

$$T^H \sum_1^n n^{-1} (B_k^H - B_{k-1}^H) (B_{k-1}^H - B_{k-2}^H) \not\xrightarrow{pr} 0$$

since by Birkhoff's ergodic Thm. We have

$\sum_1^n \square / n \xrightarrow{a.s.} \mathbb{E}\left((B_2^H - B_1^H)(B_1^H - B_0^H) \mid \mathcal{I}\right)$

But it has nonzero expectation.

④ <u>Zeros of 1-dim BMs:</u>

Set $\mathcal{Z}(w) := \{t \geq 0 \mid B_t(w) = 0\}$. $B_t \in \mathbb{R}^1$.

It's easy to see $\mathcal{Z}(w)$ is closed (conti.)

and unbdd (recurrent.)

Set $\tau_q \stackrel{\Delta}{=} \{t \geq q \mid B_t(w) = 0\}$. $q \in \mathbb{Q}^+$.

By SMP: $\mathbb{P}_1(T_0 \circ \theta_{\tau_q} = 0 \mid \mathcal{F}_{\tau_q}) = \mathbb{P}_0(T_0 = 0) = 1$

$\Rightarrow \mathbb{P}\left(\bigcap_{\mathbb{Q}^+} \{T_0 \circ \theta_{\tau_q} = 0\}\right) = 1$. $(*)$

1°) Actually, $\mathcal{Z}$ has no isolated point:

$\forall t \in \mathcal{Z}$. if $(t, t+\varepsilon) \cap \mathcal{Z} = \emptyset$. $\forall \varepsilon > 0$. then:

$\forall \tilde{\varepsilon} > 0$. $\exists q \in (t - \tilde{\varepsilon}, t)$. $q \leq \tau_q \leq t$.

(If $t = \tau_q$. By $(*)$, τ_q can't be isolated from right. contradict! So: $q \leq \tau_q < t$. $\tau_q \neq t \to t$)

2) $\mathcal{Z}(w)$ is uncountable.

<u>Thm.</u> $\forall$ complete metric space X has finite isolated points is uncountable. $(*)$

Pf: By contradiction: Note X is Baire space

Set $\tilde{X} = X / \{i_k\}_1^\infty$ is still complete

and metric space so Baire. Since

i_k's are isolated.

But set $G_n = \tilde{X} / \{x_n\}$. Where $\tilde{X} = \{x_n\}$

G_n dense in $\tilde{X}$. But $\cap G_n = \emptyset$.

Rmk: i) $Z(w)$ is perfect set as Cantor set.

ii) $(*)$ is false if X has countable

points. e.g. $X = \{x_k\}_1^N$, $N \in \mathbb{Z}^+$. x_k

are all isolated. $d(x_k, x_j) = 1 - \delta_{kj}$

⑤ Hitting time:

Lemma. $T_a := \inf\{t \geq 0 \mid f(t) = a\}$ is left-conti

and right limit exists. for $f \in C(\mathbb{R}^{\geq 0}, \mathbb{R}^1)$ and $f(0) = 0$.

Pf: $(T_a) \uparrow$. So right-limit exists

$\forall s < T_a$. $\sup_{t \leq s} f(t) = b < a$.

Set $\tilde{a} \in (b, a)$. We have $T_{\tilde{a}} \in (s, T_a)$. So T_a is left-conti.

Set $\tau_a(\omega) := \inf\{t \geq 0 \mid B_t(\omega) = a\}$

So the properties in Lemma hold for $\tau_a(\omega)$

Besides, (τ_a) is a.s. right conti.

Pf: First note that by SMP:

$\tau_a - \tau_b \overset{\mathcal{L}}{\sim} \tau_{a-b}$. Next. check: $\mathbb{P}(\tau_{0+} = 0) = 1$

$\mathbb{P}(\tau_{0+} < \varepsilon) \overset{mono}{=} \mathbb{P}(\bigcup\{\tau_{1/n} < \varepsilon\})$

$\geq \mathbb{P}(\tau_{\varepsilon^2} < \varepsilon)$

$\geq \mathbb{P}(B_{\varepsilon^2} > \varepsilon) = \mathbb{P}(B_1 > 1) > 0$

By Blumenthal 0-1 law. $\{\tau_{0+} = 0\} = \bigcap\{\cdots < \varepsilon\}$

⑥ B_t only $(\frac{1}{2} - \varepsilon)$-Hölder on cpt interval.

Actually $\sup\limits_{0 \leq s < t \leq \infty} |B_{t,s}| / |t-s|^{\beta} = +\infty$, $\forall \beta \in (0,1]$

Pf: $t B_{\frac{1}{t}} \sim B_t$. $B_{t,s} \sim B_{s-t}$. So. we can just consider $\beta \in (0, \frac{1}{2}]$.

With it provided from LIL.

⑦ To define general $(\mathcal{G}_t)$-BM. (B_t). where $\mathcal{G}_t \neq \mathcal{G}_t^{B}$. We require $B_{t+s} - B_s$ is indept of $\mathcal{G}_s$ & B_t is $\mathcal{G}_t$-adapted to have SMP.