(c) Nontrivial conti. local mart. has infinite variation. (But for discconti. case. It doesn't hold. e.g. $N_t - \lambda t$)

Rmk: i) Due to it, we can't integrate general conti. process on c.l.m M_t.

e.g. $\exists\ H_t^{(n)}(\omega)$ conti. s.t. $|H_t^{(n)}(\omega)| \leq 1$.

And $\sum_0^{m_n} H_{\tilde{t}_k}^{(n)} B_{\hat{t}_k, \hat{t}_{k+1}} = \sum |B_{\hat{t}_k, \hat{t}_{k+1}}| \to \infty$

We rather integrate progressive process and define it as a L^2-limit. (not point-wise defined!)

ii) We can also integrate on càdlàg semimart.

L^p-bdd c.l.m may also not be a true mart.! (e.g. $d=3$. $M_t = 1/|B_{t+1}|$. $\mathcal{G}_t = \mathcal{F}_{1+t}$.

We have $\mathbb{E}(1/|B_t|^2) = 1/t$. So M_t is L^2-bdd. But B_t on $\mathbb{R}^3$ is transient.

So $|B_t| \xrightarrow{a.s.} \infty$. if M_t is a mart. Then $M_t = \mathbb{E}(M_\infty | \mathcal{G}_t) = 0$ since $M_\infty = \lim M_t = 0$.)

Rmk: i) L^p-bdd of $M_t \Rightarrow (M_t)_t$ is u.i. But $(M_{\tau_n \wedge t})_n$ may not be u.i.! (e.g. consider on $[0,\infty)$ by replacing $[0,t)$. $M_x = x$)

ii) Doob's inequality doesn't work on c.l.m. So we can't deduce $\sup_t |M_t|^p \in L^1$. We can't find a real dominant!

Pf: i) $\mathcal{H}^{p.c} := \{ M_t$ is conti. mart. $| \ \|M\|_{\mathcal{H}^{p.c}} := \sup_{t\geq 0} \|M_t\|_p < \infty \}$. for $p > 1$.

Rmk: i) $\forall M \in \mathcal{H}^{p.c}$. $\sup_{t\geq 0} \|M\|_p = \|M_\infty\|_p \sim \|\sup_{t\geq 0} M_t\|_p$.

ii) $\mathcal{H}^{p.c}$ is Banach space. $\forall (M^n)$ Cauchy in $\mathcal{H}^{p.c}$. $\Rightarrow M^n_\infty \xrightarrow{L^p} M_\infty$

So: $\mathbb{E}(M^n_\infty | \mathcal{F}_t) = M^n_t$

$\longrightarrow \mathbb{E}(M_\infty | \mathcal{F}_t) \stackrel{\Delta}{=} M_t$.

Since $M^n_t \to M_t$ in L^p. With

Doob's. We conclude $M^n \xrightarrow{ucp} M$. So we can find conti modifi. of M. s.t. $M^n_\infty \xrightarrow{L^p} \tilde{M}_\infty$

ii) $(X^n_t)_t \to (X_t)$ uniformly in Prob. (ucp) if $\sup_{[0,T]} |X^n_t - X_t| \xrightarrow{pr} 0$. for $\forall T > 0$.

Prop. $\mathcal{X}^c := \{$ conti. stoc. process $\}$. with $D^{ucp}(X, Y) := \mathbb{E}(D(X,Y)) := \mathbb{E}(\sum_{k=1} 2^{-k} \wedge \sup_{[0,k]} |X_t - Y_t|)$ is complete.

Lemma D^{ucp} induces ucp convergence.

Pf: $\forall k$. $\mathbb{P}(1 \wedge \sup_{[0,k]} |X_t - Y_t| \geq \varepsilon)$
$\leq \mathbb{E}(\sup_{[0,k]} |X_t - Y_t| \wedge 1) \leq 2^k D_{ucp}(X, Y)$

Conversely. $X^n \xrightarrow{ucp} X$. By DCT. We have $D^{ucp}(X_n, X) \to 0$.

Pf: D. D^{ucp} are both metric. And D just induces uniform converge on cpt set a.s. (Weaks).

For (X^n) Cauchy in D^{ucp}.

$\exists (n_j)$. s.t. $\sum D^{ucp}(X^{n_j}, X^{n_{j+1}}) \leq \sum 2^{-j} < 1$

$\Rightarrow (X^{n_j}(\omega))$ is $\overset{a.s.}{\text{Cauchy}}$ in $(C(\mathbb{R}^{\geq 0}), D)$

So: $\exists X \in C(\mathbb{R}^+)$. s.t. $D(X^{n_j}, X) \xrightarrow[a.s.]{j\to\infty} 0$

By DCT. $D^{ucp}(X^{n_j}, X) \xrightarrow{j\to\infty} 0$

With triangular inequi. $D^{ucp}(X^n, X) \to 0$

Cor. $(X^n) \xrightarrow{ucp} X \Rightarrow \exists (n_j)$. $X^{n_j} \to X$

uniformly on cpt sets. a.s.

Prop. i) $(M^n) \subset \mathcal{M}^c_{loc} \xrightarrow{ucp} M$. $\Rightarrow M \in \mathcal{M}^c_{loc}$

ii) $(M^n) \subset \mathcal{M}^c_{loc}$. We have $M^n \xrightarrow{ucp} M \Leftrightarrow$

$\langle M^n - M \rangle \xrightarrow{ucp} 0$.

Rmk: ucp limit of càdlàg mart may not even be local mart. It needs local integrability cond. on jumps:

$\sup_n \sup_{s \leq t} |\Delta M^n_s| \in L^1_{loc}(\sim)$. for $M \in \mathcal{M}_{loc}$

Pf: ii) Using Lenglart inequi. Set $a = \varepsilon$

$b = \varepsilon^2$. on $(M^2, \langle M \rangle_t)$.

Rmk: It may not hold for conti. semimart since we also use BDG.

⑧ Notes on BDG:

Def: (X_t), $(A_t) \geq 0$, progressive. X is Lenglart dominated by A if $\forall$ bdd stopping time τ, we have $\mathbb{E}(X_\tau) \leq \mathbb{E}(A_\tau)$.

Thm. (Lenglart's inequi.)

$X \geq 0$, conti. adapted, is Lenglart domi. by $A \geq 0$, $\uparrow$, conti. adapted. Then:

i) $\forall a, b > 0$, $\mathbb{P}(X_t^* \geq a) \leq \frac{1}{a}\mathbb{E}(A_t^* \wedge b) + \mathbb{P}(A_t^* > b)$.

ii) $\forall p \in (0,1)$, $\mathbb{E}((X_\infty^*)^p) \leq \frac{p^{-p}}{1-p}\mathbb{E}(A_\infty^p)$.

Pf: i) Set $\tau = \inf\{s \in [0,t] \mid A_s \geq b+\varepsilon\}$

$\tilde{\tau} = \inf\{s \in [0, t\wedge\tau] \mid X_s \geq a-\varepsilon\}$

$LHS \leq \mathbb{P}(X_t^* \geq a, \tau > t) + \mathbb{P}(\tau \leq t)$

$\overset{\text{mono.}}{\leq} \mathbb{P}(X_{t\wedge\tau}^* \geq a-\varepsilon) + \mathbb{P}(A_t^* > b)$

$\overset{\text{chebyshev}}{\leq} \int_{\{X_{t\wedge\tau}^* \geq a-\varepsilon\}} X_\square \, d\mathbb{P} / a-\varepsilon + \text{⊟}$

Domi:
$\leq \mathbb{E}(G_{\tau\wedge t}) / \lambda - \varepsilon + \square$. Set $\varepsilon \to 0$.

ii) Note $\mathbb{E}(X^p) = \int_0^\infty p x^{p-1} P(X > x)\, dx$

choose $a = x^{\frac{1}{p}}$, $b = x^{\frac{1}{p}} / p$.

Rmk: It can be used to prove BDG inequi. when $p \in (0, 1)$ from:

$\mathbb{E}(M_\tau^2) = \mathbb{E}(\langle M \rangle_\tau)$.

Set $X_t = M_t^2$, $G_t = \langle M \rangle_t$ and

$X_t = \langle M \rangle_t$, $G_t = \sup_{s \leq t} M_s^2$.

But it doesn't apply on disconti. mart. (Recall in this case, we only have $p \geq 1$).

④ By Tanaka's formula: Semimart can be retained in difference of convex func.

Rmk: $\forall \alpha \in (0, \frac{1}{2})$, $\forall$ local mart X_t, $|X_t|^\alpha$ isn't a semimart unless X_t is trivial

(5) Recall discrete mart with bdd increment either $\lim_{n\to\infty} M_n$ exist finite a.s. or hits every pt of $\mathbb{R}^1$.

In case of conti. mart. Then, a.s.:

Either i) $\lim_{t\to\infty} M_t$ exist. finite.

Or ii) $\overline{\lim_{t\to\infty}} M_t = +\infty$. $\underline{\lim} M_t = -\infty$.

Rmk: i) Note for $M_t \geq 0$. conti. mart.

since $\sup \mathbb{E}(M_t^-) = 0 < \infty$. $\Rightarrow$

$\lim M_t$ exists finite. a.s. satisfies the case above!

ii) For conti. submart or supermart.

$\mathbb{P}(\{\lim M_t \text{ exists finite}\} \cup \{\overline{\lim} M_t = +\infty\})$

or $\mathbb{P}(\{\lim M_t \text{ exists finite}\} \cup \{\underline{\lim} M_t = -\infty\})$

$= 1$.

iii) Recall also. for $M \in \mathcal{M}_c^{loc}$

$\{\lim_{t\to\infty} M_t \text{ exists. finite}\} \overset{a.s}{=} \{\langle M\rangle_\infty < \infty\}$

⑥ L^1-bdd backward submart. $(X_t)_{\mathbb{R}\le 0}$ converges a.s. & L^1.

Pf: 1) By upcrossing inequi. $X_t \xrightarrow{a.s.} X_{-\infty} \in L^1$.

2) $Y_t := \mathbb{E}(X_0 | \mathcal{G}_t) - X_t$ is positive supermart. Next. we prove (Y_t) u.i.

$$\int_{\{Y_t \le k\}} Y_t \, d\mathbb{P} \ge \int_{\{Y_t \le k\}} Y_s \, d\mathbb{P}.$$

for $\forall\ t \le s \le 0$

$$\Rightarrow \int_{\{Y_t \ge k\}} Y_s \, d\mathbb{P} + \mathbb{E}(Y_t - Y_s) \ge$$

$$\int_{\{Y_t \ge k\}} Y_t \, d\mathbb{P}.$$

$(\mathbb{E}(Y_s))$ $\uparrow$. bdd. $\Rightarrow$ limit exist.

Fix s. st. $\mathbb{E}(Y_s - Y_r) \le \varepsilon$. $\forall r \le s$.

$\mathbb{P}(Y_t \ge k) \le \sup \mathbb{E}(Y_t)/k$.

$\exists \tilde{k}$. st. $\forall k \ge \tilde{k}$. $\mathbb{P}(Y_t \ge k) < \delta$.

$$\Rightarrow \int_{\{Y_t \ge k\}} Y_s \, d\mathbb{P} < \varepsilon.$$

⑦ Realization of conti. stochastic process on $(C(\mathbb{R}^{\ge 0}), \mathcal{B}(C(\mathbb{R}^{\ge 0})))$. where $\mathcal{B}(\mathbb{R}^{\ge 0}) =$

$\mathcal{B}(\mathbb{R})^{\mathbb{R}^+} \cap C(\mathbb{R}^+) = \sigma((W_t)_{t\geq 0})$:

For $X_t \in \mathbb{R}^1$. conti. SP. with law $\mathbb{P}_X = \mathbb{P} \circ X^{-1}$. (W_t) coordinate func. on $(C(\mathbb{R}^{\geq 0}),\ \mathcal{B}(C(\mathbb{R}^{\geq 0})).\ \mathbb{P}_X)$ has same law as (X_t).

<u>Pf</u>: $(W_t)_t$ is conti. measurable from $(C(\mathbb{R}^{\geq 0}),\ \mathcal{B}(C(\mathbb{R}^{\geq 0})))$ to $(\mathbb{R}^1, \mathcal{B}_{\mathbb{R}^1})$.

Denote $\Gamma := \{ f \in \mathbb{R}^{\mathbb{R}^{\geq 0}} \mid f(t_i) \in A_i \}$.

$\mathbb{P}_X(\{ f \in C(\mathbb{R}^{\geq 0}) \mid W_{t_i}(f) \in A_i \}) =$

$\mathbb{P}_X((W_t)_{t\geq 0} \in \Gamma \cap C(\mathbb{R}^{\geq 0})) =$

$\mathbb{P}(X \in \Gamma \cap C(\mathbb{R}^{\geq 0})) = \mathbb{P}(X_{t_i} \in A_i)$

<u>Rmk</u>: Let $X = B$. B is BM $\Rightarrow$ $\mathbb{P}_B$ is Wiener measure & (W_t) is BM on $C(\mathbb{R}^{\geq 0}).\ \mathcal{B}(C(\mathbb{R}^{\geq 0}))$

⑧ Doob's inequi. when $p=1$:

If X is càdlàg mart. Then:

$$E(\sup_{[0,T]} |X_t|) \leq \frac{e}{e-1}(1 + E(|X_T| \log |X_T|))$$

⑨ About r.c.p. : (Motivation)

r.c.p is to generalize cond. prob. $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$

Note : $\mathbb{P}(A) = \sum_{x \in \mathcal{X}} \mathbb{P}(A \mid X = x)\, \mathbb{P}(X = x)$ can happen. i.e. uncountable sum of null measure sets yields a positive probability.

So we need to introduce r.c.p. as a disintegration. e.g. μ is r.c.p w.r.t $\sigma(Y)$.

$$E(f(X,Y) \mid \sigma(Y))(\omega) \overset{a.s.}{=} \int f(x, Y(\omega))\, d\mu(\omega, x)$$

Rmk : i) r.c.p. has "regular" prop. at cost of assumption on topo. space. (general case : value space is polish)

ii) r.c.d. is only defined on range of some r.v. (Push forward r.c.p. by some r.v. can also get r.c.d.)

⑩ Why choose progressive :

i) Assumpt of continuity path & adapted is too demanding.

ii) To define sto-integral well:

Progressive can be inherited in sto-integration. (Also can be proved: it's closed under $\max_{s \le t} X_s$ / $\min_{s \le t} X_s$)

Also, note: $H^n := \sum_k \mathrm{sgn}(B_{t_k^n, t_{k+1}^n}) I_{(t_k^n, t_{k+1}^n]}$ isn't progressive. Then: $\int H^n dB = \sum_k |B_{t_k^n, t_k^{n+1}}| \xrightarrow{n \to \infty} \infty$. So the integral is not stable under limit.

iii) We use predictable process (generated by left-conti. & adapted, i.e. info. can only be known from past side) $\sum I_{(s,t]} h(w)$ to approxi. $\int H dM$

It's because when H & M are càdlàg $H \cdot M$ may jump at the same time, which can be problematic: $H \cdot M$ may not be a local mart. e.g.

$H = N_t$ poisson process & $M = N_t - t$. But

$$\int N_s\, dM_s = \int_0^t N_{s-}\, dM_s + \sum (\Delta N_s)^2$$

$$= \int N_{s-}\, dM_s + N_t .$$

Note $\int_0^t N_{s-}\, dM_s \in \mathcal{M}_{loc}$, but $N_t \notin \mathcal{M}_{loc}$

<u>Rmk</u>: Using pred. process to approximate can preserve its mart. or local mart. property. Note that in the case of Stratonovich integral. it ruin the randomness property. but it has good analysis property:

$df(X_t) = \sum_1^d \partial_i f(X_t) \circ dX_t^i$ for $f \in C^3(\mathbb{R}^d; \mathbb{R}^1)$. X_t^i is semimart.

And it has Wong-Zakai principle:

$(X^n) \subset FV \xrightarrow{ucp} X$ conti. semimart.

$\Rightarrow \int f(X^n)\, dX^n \xrightarrow{ucp} \int f(X) \circ dX$. for $f \in C^2$. (Set $F' = f$. So it becomes to prove $F(X^n) \xrightarrow{ucp} F(X)$. which's done by subseq argument.)