# Itô's Integral

## (1) Itô's Integral:

Consider SDE: $dX/dt = b(t, X_t) + \sigma(t, X_t) \cdot W_t$.

Where $W_t$ represents noise term.

Base on some requirement of Engineer:

We assume: i) $W_{t_1}$ indept with $W_{t_2}$. if $t_1 \neq t_2$.

ii) $W_t$ is stationary.

iii) $\mathbb{E}(W_t) = 0$.

Rmk: Actually such $(W_t)$ isn't reasonable:

It's not conti. a.s. when satisfying i) ~ iii).

Pf: Set $W_t^{(n)} = (-N) \vee (N \wedge W_t)$. truncated.

$\mathbb{E}((W_t^{(n)} - W_s^{(n)})^2) \xrightarrow{n \to \infty,\ t \to s} 0$ only when

$Var(W_t) = 0$. i.e. $W_t = 0$. a.s.

If require $\mathbb{E}(W_t) = 1$. Then $W_t(w) \notin \mathcal{B}_{\mathbb{R}_{\geq 0}} \times \mathcal{F}$.

which is more pathological.

Next, we will represent $(W_t)$ as a generalized process (It even exists $\mathbb{P}$ on $S^*[0, \infty)$):

Rewrite the SDE : $X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k) W_k \Delta t_k$.

where $X_k = X(t_k)$. $W_k = W(t_k)$. $\Delta t_k = t_{k+1} - t_k$.

Def: $V_{t_{k+1}} - V_{t_k} =: \Delta V_k = W_k \Delta t_k$.

Rmk: $(V_t)_{t\geq 0}$ is stationary. indept increment with mean 0.

Thm. If $V_t$ has conti. path. Then $V_t = B_t$. a.s.

$$\Rightarrow X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j.$$

A natural idea:

Set $\Delta t \to 0$. Express in integration notation.

However. TV of $B_t$ is too big to define Rieman - Stieltjes integral. which will depend on choice of partition points:

Def: To approxi $f(t,w)$ in $\int_S^T f(t,w)\, dB_t(w)$.

we consider use $\sum_j f(t_j^*, w) \chi_{[t_j, t_{j+1})}(w)$.

s.t. $t_j^* \in [t_j, t_{j+1})$.

i) $t_j^* = t_j$. It leads to Itô integral.

ii) $t_j^* = \frac{t_{j+1} + t_j}{2}$. It leads to Stratonovich integral. And we denote i) by $\int_S^T f\, dB_t$.

ii) by $\int_S^T f \circ dB_t$

Rmk: i) means "Not looking into Future". while ii) has advantage in connecting with SDE on manifolds by its form.

① Construction:

Def: $\mathcal{V} = \mathcal{V}(S,T) =: \{ f(t,w) : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{R}^1 \mid$ satisfy i) ~ iii)

i) $(t,w) \mapsto f(t,w) \in \mathcal{B}_{\mathbb{R}_{\geq 0}} \otimes \mathcal{F}_\infty^B$

ii) $f(t,w) \in \mathcal{F}_t^B$ iii) $\mathbb{E}(\int_S^T f^2\, dt) < \infty$. $\}$.

Lemma. $\phi(t,w)$ bdd elementary process. Then:

$\mathbb{E}((\int_S^T \phi\, dB_t)^2) = \mathbb{E}(\int_S^T \phi^2\, dt)$.

where $\phi = \sum e_j(w) \chi_{[t_j, t_{j+1})}$. and $\int \phi\, dB_t$ defined by $\sum e_j(w)(B_{t_{j+1}}(w) - B_{t_j}(w))$.

Step. 1. $g \in \mathcal{V}$ bdd. conti. for each $w \in \Omega$.

$\exists (\phi_n) \in \mathcal{V}$. elementary. $\mathbb{E}(\int_S^T (g - \phi_n)^2) \to 0$.

Pf: Direct by conti. and BCT.

<u>Step. 2.</u> $h \in V$ bdd.

$\exists\, g_n \in V$. bdd conti. $\forall w \in n$.

s.t. $\mathbb{E}\left(\int_s^T (g_n - h)^2\right) \to 0$.

<u>Pf:</u> $g_n = h * e_n$. $(e_n)$ mollifiers.

Apply BCT. for $n \to \infty$.

<u>Step. 3.</u> $f \in V$.

$\exists\, h_n \in V$. bdd. $\mathbb{E}\left(\int_s^T (f - h_n)^2\right) \to 0$.

<u>Pf:</u> Set $h_n = -n \vee (f \wedge n)$

Apply DCT.

<u>Def:</u> For $f \in V(S,T)$. Then: $\int_s^T f(t,w)\, dB_t(w)$

$=: \lim_n \int_s^T \phi_n(t,w)\, dB_t(w)$. in $L^2(P)$.

where $(\phi_n)$ is seq of elementary func

s.t. $\mathbb{E}\left(\int_s^T (f - \phi_n)^2 dt\right) \to 0$.

<u>Rmk:</u> We have Itô isometry : $\|f\|_{M^2}$

$= \|f\|_{L^2(B)}$. $\forall f \in V(S,T)$.

<u>prop.</u> If $f, f_n \in V(S,T)$. $\mathbb{E}\left(\int_s^T (f_n - f)^2\right) \to 0$

Then $\int_s^T f_n\, dB_t \xrightarrow{L^2(P)} \int_s^T f\, dB_t$.

② **Properties:**

Thm. $f, g \in V(0,T)$. Set $0 \le S < u < T$. Then:

i) $\int_S^T c(f+g) = c\int_S^T f\,dB_t + \int_S^T g\,dB_t$.

ii) $\mathbb{E}(\int_S^T f\,dB_t) = 0$.

iii) $\int_S^T f\,dB_t \in \mathcal{F}_T^B$.

Thm. For $f \in V(0,T)$. Then $\int_0^t f\,dB_s$ has a $t$-conti. modification $J_t$. $\forall\, 0 \le t \le T$.

Pf: $\exists\, \phi_n$ elementary $\to f$ in $H^2$.

Set $I_n = \int_0^t \phi_n\,dB_s$. conti.

1°) $I_n$ is a mart. w.r.t $\mathcal{F}_t$.

2°) Apply Doob's inequality:

$M_t$ is right-conti. mart. $\forall p \ge 1$. $T \ge 0$.

$\lambda > 0$. $P(\sup_{[0,T]} |M_t| \ge \lambda) \le E|M_T|^p / \lambda^p$.

$\Rightarrow \exists\, (I_{n_k})$ uniformly converges in $[0,T]$. Set the limit is $J_t$.

Cor. For $f \in V(0,T)$. $\forall T$. Then: $M_t = \int_0^t f(s,\omega)\,dB_s(\omega)$ is mart. w.r.t $\mathcal{F}_t$

Rmk: As for Stratonovich integral. $\int_0^t f \circ dB_s$ isn't mart.

③ <u>Extension:</u>

First. modify the measurable condition ii):

ii*) $\exists\ \mathcal{H}_t \uparrow$ $\sigma$-algebra. s.t.

$f_t \in \mathcal{H}_t$. $(B_t)$ is mart. w.r.t $(\mathcal{H}_t)$.

<u>Rmk:</u> It implies $\mathcal{F}_t \subset \mathcal{H}_t$.

Then. we can apply on $(\vec{B}_t) = (B_t^1 \cdots B_t^n)$. Set $\mathcal{H}_t = \sigma(B_s^i.\ 0 \le s \le t.\ 1 \le i \le n)$. So: $\int_0^t f(s,w)\, dB_s^k$ is legal. e.g. $\int \sin(B_t^1 + B_t^2)\, dB_t^2$

<u>Def:</u> $\vec{B}_t$ is n-dim BM. Set $V_H^{m\times n}(S,T) =$

$\{ V = (V_{ij})_{m\times n} \mid V_{ij}$ satisfies. i). ii*). iii) $\}$

For $V \in V_H^{m\times n}$. $\int_S^T V\, dB = \int_S^T \begin{pmatrix} V_{11} & \cdots & V_{1n} \\ & \vdots & \\ V_{m1} & & V_{mn} \end{pmatrix} \begin{pmatrix} dB^1 \\ \vdots \\ dB^n \end{pmatrix}$

<u>Rmk:</u> i) $m=1$. Denote $V_H^{1\times n} = V_H^n$.

ii) $V_H^{m\times n}[0,\infty) = \bigcap_{T>0} V_H^{m\times n}[0,T]$.

Second. modify condition iii):

iii*) $P(\int_S^T f^2(s,w)\, ds < \infty) = 1$.

<u>Def:</u> For $(\vec{B}_t)$. n-dim BM. Set $W_H^{m\times n}(S,T)$

$= \{ V \in M_{m\times n} \mid V_{ij}$ satisfies i). ii*). iii*) $\}$.

Denote: $\mathcal{V}_n = \mathcal{V}$, $\mathcal{W}_n = \mathcal{W}$, if: $\mathcal{H}_t = \sigma(B_s^k,\ 0 \le s \le t,\ 1 \le k \le n)$

Rmk: Actually, we can prove:

For $f \in \mathcal{W}_n$, $\forall t$, $\exists f_n \in \mathcal{W}_n$ st. $\int_0^t |f_n - f|^2 \to 0$ in prob. $(f_n)$ is seq of step function.

So Define: $\int_0^t f \, dB_s = \lim_n \int_0^t f_n \, dB_s$ in pr.

But it's local mart, rather than mart.

Prop. $\exists$ t-conti version of it as well.

(2) Itô Process:

Def: $(B_t)$ is 1-dim BM on $(\Omega, \mathcal{F}, \mathbb{P})$. A 1-dim Itô process is $X_t = X_0 + \int_0^t u(s,\omega) \, ds + \int_0^t v(s,\omega) \, dB_s(\omega)$, where $v \in \mathcal{W}_n$.

Rmk: Sometimes we write in form:

$$dX_t = u \, dt + v \, dB_t.$$

Thm. (1-dim Itô Formula)

$dX_t = u\,dt + v\,dB_t$, Itô process. For $g(t,x) \in C^2(\mathbb{R}_{\ge 0} \times \mathbb{R}^1)$, $Y_t = g(t, X_t)$ is a Itô process again. $dY_t = \frac{\partial g}{\partial t}(t, X_t)\,dt + \frac{\partial g}{\partial x}(t, X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2$.

**Rmk:** If $X_t(\omega) \in U$. $\forall t, \omega$. Then

it's enough $g \in C^2([0,\infty) \times U)$

**Cor.** (Integrate by part)

$f$ is conti. of BV on $[0,t]$. a.s. $\omega$.

Then. $\int_0^t f(s)\, dB_s = f(t) B_t - \int_0^t B_s\, df_s$

**Def:** For $\vec{B}_t$ n-dim BM. n-dim Itô-process

$\vec{X}_t$ is $d\begin{pmatrix} X_t^1 \\ \vdots \\ X_t^n \end{pmatrix} = u\,dt + v\,d\vec{B}_t$. where

$v = (v_{ij})_{m\times n} \in W_n^{m\times n}$ $u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in W_n^m$.

**Thm** (n-dim Itô Formula)

$dX_t = u\,dt + v\,dB_t$. $g(t,x) = (g_1(t,x) \cdots g_p(t,x))$

$\in C^2(\mathbb{R}_{\geq 0} \times \mathbb{R}^n, \mathbb{R}^p)$ Then. $Y(t, X_t) = g(t, X_t)$.

is p-dim Itô Process again. st.

$$Y_k(t) = \frac{\partial g_k}{\partial t} dt + \sum_i \frac{\partial g_k}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j} dX_i\, dX_j$$