Diffusion Theory

To describe motion of a small particle suspend in a moving liquid, subject to random molecular bombardments. If $b(t,x)$ is velocity of fluid at time t and point x. Establish SDE:

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t) W_t.$$

Intepret in Itô:

$$dX_t = b(t, X_t) + \sigma(t, X_t) dB_t$$

Rmk: b is drift coefficient and σ is diffusion coefficient

Def: (Time-homogeneous) Itô diffusion is a stochastic process $X_t(\omega) : [s, \infty) \times \Omega \to \mathbb{R}^n$. satisfies: $dX_t = b(X_t) dt + \sigma(X_t) dB_t$, $t \geq s$ and $X_s = x$.

where $\vec{B_t}$ is m-dim BM. $b : \mathbb{R}^n \to \mathbb{R}^n$. $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ satisfies Lipschitz condition:

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x-y|.$$

Rmk: i) In fact, we can conclude X_t is strong Feller process by Lipschitz.

ii) Note it satisfies E & U Thm. for any bdd interval. Denote the unique solution by $X_t = X_t^{s,x}$, $t \geq s$.

iii) Replace $\sigma(X_t)$, $b(X_t)$ by $\sigma(X_t, t)$, $b(X_t, t)$, it becomes inhomogeneous case.

(1) Properties:

① Time-homogeneous:

prop. $(X_t)_{t\geq 0}$ is time-homogeneous.

Pf: $X_{s+h}^{s,x} = x + \int_s^{s+h} b(X_u^{s,x}) du + \int_s^{s+h} \sigma(X_u^{s,x}) dB_u$

$\overset{u=s+v}{=} x + \int_0^h b(X_{s+v}^{s,x}) dv + \int_0^h \sigma(X_{s+v}^{s,x}) d\tilde{B}_v$

where $\tilde{B}_v = B_{s+v} - B_s$.

$X_h^{0,x} = x + \int_0^h b(X_v^{0,x}) dv + \int_0^h \sigma(X_v^{0,x}) d\tilde{B}_v$

also satisfies identical SDE.

By strong Uniqueness:

$(X_{s+h}^{s,x})_{h\geq 0} \overset{d}{\sim} (X_h^{0,x})_{h\geq 0}$

Rmk: Moreover, we have $(X_{s+h}^{s,x})_{h\geq 0}$ indept with $\sigma(B_r, r \leq s)$.

② Markov Property:

Denote: i) Q^x is law of $(X_t^{0,x})$.

ii) $\mathcal{F}_t^{(m)} = \sigma(B_s,\ s \le t)$. $\mathcal{M}_t = \sigma(X_s,\ s \le t)$

Rmk: $\mathcal{M}_t \subset \mathcal{F}_t^{(m)}$. by E & U. Thm.

Thm. (Simple Markov)

f is bdd Borel $: \mathbb{R}^n \to \mathbb{R}^1$. Then $\forall t, h \ge 0$.

$$\mathbb{E}^x (f(X_{t+h}) \mid \mathcal{F}_t^{(m)})(\omega) = \mathbb{E}^{X_t(\omega)} (f(X_h))$$

Pf: $F(x,t,r,\omega) \triangleq X_r^{t,x}(\omega)$. $r \ge t$. indept of $\mathcal{F}_t^{(m)}$

$\Rightarrow X_r(\omega) = F(X_t, t, r, \omega)$

prove: $\mathbb{E}(f(F(X_t, t, t+h, \omega)) \mid \mathcal{F}_t^{(m)}) =$

$\mathbb{E}(f(F(x, 0, h, \omega))) \mid_{x = X_t(\omega)}$

Set $g(x,\omega) = f \circ F(x, t, t+h, \omega)$

$\exists \sum \phi_k(\omega) \varphi_k(x) \nearrow g(x,\omega)$.

$\mathbb{E}(\sum \phi_k(\omega) \varphi_k(X_t) \mid \mathcal{F}_t^{(m)}) =$

$\sum \varphi_k(X_t) \mathbb{E}(\phi_k(\omega) \mid \mathcal{F}_t^{(m)}) =$

$\sum \varphi_k(\eta) \mathbb{E}(\phi_k(\omega) \mid \mathcal{F}_t^{(m)}) \mid_{\eta = X_t}$

$\Rightarrow \mathbb{E}(g(X_t, \omega) \mid \mathcal{F}_t^{(m)}) = \mathbb{E}(g(\eta,\omega) \mid \mathcal{F}_t^{(m)}) \mid_{\eta = X_t}$

$= \mathbb{E}(g(\eta,\omega)) \mid_{\eta = X_t}$. by indept!

Finally, apply time-homogeneous of (X_t).

Rmk: Since $\mathcal{M}_t \subset \mathcal{F}_t^{(m)}$. So (X_t) is also $\mathcal{M}_t$ - Markov process.

Thm (Strong Markov)

$f: \mathbb{R}^n \to \mathbb{R}^1$ bdd. Borel. τ is stopping time w.r.t $\mathcal{F}_t^{(m)}$. s.t. $\tau < \infty$. a.s. Then:

$$\mathbb{E}^x (f(X_{\tau+h}) \mid \mathcal{F}_\tau^{(m)}) = \mathbb{E}^{X_\tau(w)} (f(X_h))$$

Pf: Set $F(x,t,r,w) = X_r^{t,x}(w)$. $r \geq t$. again,

Argue as in Time-homo. by Strong unique:

$F(X_\tau, \tau, \tau+h, w)$ indept with $\mathcal{F}_\tau^{(m)}$.

prove: $\mathbb{E}(f(F(x,0,\tau+h,w)) \mid \mathcal{F}_\tau^{(m)}) = \mathbb{E}(f(F(x,0,h,w)))\Big|_{x = X_\tau^{0,x}}$

Similar as before: $\sum \phi_k(x) \psi_k(t,r,w) \uparrow g(x,t,r,w)$

$=: f(F(x,t,r,w))$.

Cor. $(f_k)_1^n$ are bdd. Borel on $\mathbb{R}^n$. Then:

$$\mathbb{E}^x (\prod_1^n f_i(X_{\tau+h_i}) \mid \mathcal{F}_\tau^{(m)}) = \mathbb{E}^{X_\tau} (\prod_{k=1}^n f_k(X_{h_k}))$$

Pf: By induction. directly.

Cor. $\forall \eta \in \mathcal{M}_\infty$. bdd. Then:

$$\mathbb{E}^x (\theta_\tau \eta \mid \mathcal{F}_\tau^{(m)}) = \mathbb{E}^{X_\tau} (\eta)$$

Pf: Approxi by Cor. above.

Rmk: In inhomogeneous case. If

i) $|\sigma(x,t)-\sigma(y,t)| \vee |b(x,t)-b(y,t)| \leq K_1|x-y|$

ii) $|\sigma(x,t)| \vee |b(x,t)| \leq K_2(|x|+1)$

Then (X_t) is strong Markov process.

③ Mean Value Property:

Denote: $H \in \mathcal{B}_{\mathbb{R}^n}$. $\tau_H^\alpha = \inf\{t>\alpha \mid X_t \notin H\}$.

where α is another stopping time.

$g \in C_B(\mathbb{R}^n)$. $\eta = g(X_{\tau_H}) I_{\{\tau_H<\infty\}}$

Lemma. $\theta_\alpha \eta \, I_{\{\alpha<\infty\}} = g(X_{\tau_H^\alpha}) I_{\{\tau_H^\alpha<\infty\}}$.

Pf: consider $\sum g(X_{t_j}) \chi_{[t_j, t_{j+1}]}(\tau_H) \to \eta$.

Cor. For $G \subset\subset H$. $G \in \mathcal{B}_{\mathbb{R}^n}$. If $\tau_H < \infty$ a.s

Then $\theta_{\tau_G} g(X_{\tau_H}) = g(X_{\tau_H})$.

Pf: $\tau_H^{\tau_G} = \tau_H$. Since $\tau_H < \infty$ a.s.

Cor. $\mathbb{E}^x(f(X_{\tau_H})) = \int_{\partial G} \mathbb{E}^y(f(X_{\tau_H}))\, Q^x(X_{\tau_G} \in dy)$

for $f \in C_B(\mathbb{R}^n)$. $G \subset\subset H$. measurable.

Pf: By cor. above. apply Markov property.

Def: Harmonic measure of X on ∂H is μ_H^x.

defined by $\mu_H^x(F) = Q^x(X_{\tau_H} \in F)$. $F \subset \partial H$.

Rmk: By above. $\phi(x) = \mathbb{E}^x(f(X_{\tau_H}))$ satisfies mean value property:

$\phi(x) = \int_{\partial H} \phi(y)\, d\mu_H^x(y)$. $\forall x \in H \subset\subset H$. Borel.

(It also holds for all time-homogeneous strong Markov process)

(2) Generator:

Def: generator of X_t is defined by:

$A f(x) = \lim_{t \to 0} \frac{\mathbb{E}(f(X_t^x)) - f(x)}{t}$ with $D(A) =$

$\{f : \mathbb{R}^n \to \mathbb{R}^1 \mid Af \text{ exists } \forall x\}$.

Lemma. $Y_t = Y_t^x(w) = x + \int_0^t u(s,w)\, ds + \int_0^t v(s,w)\, d\vec{B}_s$

is Itô process on $\mathbb{R}^n$. $\vec{B}_s$ is m-dim BM.

If $f \in C_c^2(\mathbb{R}^n)$. τ is stopping time w.r.t

$\mathcal{F}_t^{(m)}$. st. $\mathbb{E}^x(\tau) < \infty$. u, v are bdd st. Y

$\in$ supp(f). Then:

$\mathbb{E}^x(f(Y_\tau)) = f(x) + \mathbb{E}^x\left(\int_0^\tau \left(\sum_i u_i \frac{\partial f}{\partial x_i}(Y_s) + \frac{1}{2}\sum_{i,j}(vv^T)_{ij} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_s)\right) ds\right)$

Pf: Apply Itô's Formula on $f(Y_t)$.

And replace τ by $\tau_k = \tau \wedge k$ first.

Then. Let $k \to \infty$. (Converges in L^2).

Thm. For Itô diffusion $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.

If $f \in C_c^2(\mathbb{R}^n)$. Then $f \in D(A)$. St.

$$Af(x) = \sum b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Pf: By Lemma. directly check.

Cor. $A = \frac{1}{2}\Delta$. for n-dim BM. $(\vec{B_t})$.

Thm. (Dynkin's Formula)

If $f \in C_c^2(\mathbb{R}^n)$. τ is stopping time. st. $\mathbb{E}^x(\tau) < \infty$.

Then: $\mathbb{E}(f(X_\tau^x)) = f(x) + \mathbb{E}(\int_0^\tau Af(X_s^x)ds)$

Rmk: If τ is exit time of Bdd Borel set.

st. $\mathbb{E}^x(\tau) < \infty$. Then it holds for $f \in C^2$.

e.g. i) For n-dim BM: $(\vec{B_t})$. start at $\vec{a}$.

τ_k is exist time of $k = \{|x| < R\}$.

Consider $f(x) = |x|^2$. first set $\sigma_k^n = \tau_k \wedge n$

So: $\mathbb{E}^a(f(B_{\sigma_k^n})) = f(a) + \mathbb{E}^a(\int_0^{\sigma_k^n} \frac{1}{2}\Delta f(B_s)ds)$

$= |a|^2 + n\,\mathbb{E}^a(\sigma_k^n)$.

Set $r \to \infty$. LHS $\to \mathbb{E}^a(f(B_{\tau_K})) = R^2$

$\Rightarrow \mathbb{E}^a(\tau_K) = \frac{1}{n}(R^2 - a^2)$.

ii) For n-dim BM (B_t). $n \geq 2$. $|b| > R$. Next, we will find prob. of B_t start at b ever hits $K = \{|x| \leq R\}$.

Denote: τ_k is exit time of $A_k = \{R < |x| < 2^k R\}$.

$$T_K = \inf\{t > 0 \mid B_t \in K\}.$$

Set $f(x) = \begin{cases} -\log|x|, & n = 2 \\ |x|^{2-n}, & n > 2. \end{cases}$ $f_{n,k} \in C_c^2(\mathbb{R}^n)$.

St. $f_{n,k} = f$ on A_k. So $Af_{n,k} = Af = 0$ on A_k.

Denote $p_k = \mathbb{P}^b(|B_{\tau_k}| = R)$. $q_k = \mathbb{P}^b(|B_{\tau_k}| = 2^k R)$

By Dynkin's: $\mathbb{E}^b(f(B_{\tau_k})) = f(b)$. $\forall k$.

(Apply on $f_{n,k}$. Note $B_{\tau_k} \in A_k$).

1) $n = 2$:

$$-\log R \cdot p_k - \log 2^k R \, q_k = -\log|b|.$$

So: $q_k \to 0$ as $k \to \infty$.

i.e. $\mathbb{P}^b(T_K < \infty) = 1$

2) $n \geq 3$:

$$p_k \cdot R^{2-n} + q_k (2^k R)^{2-n} = |b|^{2-n}.$$

Set $k \to \infty$. $\therefore \mathbb{P}^b(T_K < \infty) = \left(\frac{|b|}{R}\right)^{2-n}$

So: BM is transient $\Leftrightarrow n \geq 3$.

Def: i) Characteristic operator A of Itô diffusion (X_t) is defined by:

$$\mathcal{A} f(x) = \lim_{k \to \infty} \frac{\mathbb{E}^x(f(X_{\tau_{U_k}})) - f(x)}{\mathbb{E}^x(\tau_{U_k})}$$

where $(U_k) \downarrow x$. open set. τ is exist time. If $\mathbb{E}^x(\tau_{U_k}) = \infty$. $\forall k$.

Then define $\mathcal{A} f(x) = 0$.

$\mathcal{D}_{\mathcal{A}} = \{ f \mid \mathcal{A} f(x) \text{ exists. } \forall x. \ \forall (U_k) \downarrow x \}$.

Rmk: We have $D_A \subseteq \mathcal{D}_{\mathcal{A}}$ and $Af = \mathcal{A} f$ on $\forall f \in D_A$.

Note that C^2 is dense in L^1. Next, we only need to prove it for C^2.

ii) $x \in \mathbb{R}^n$ is trap for (X_t) if

$Q^x(\{X_t = x. \ \forall t\}) = 1$.

Lemma. If x is not a trap for (X_t). Then:

$\exists U \ni x$. open. st. $\mathbb{E}^x(\tau_U) < \infty$.

Pf: $\exists t$. st. $Q^x(\{X_t = x\}) < 1$.

$\Rightarrow$ By conti. of X. $\exists U$ nbd of x. st. $Q^x(\{X_t \notin U\}) \triangleq p > 0$.

Discretize τ_U. $\mathbb{E}(\tau_U)$ will be dominated by expectation of $Geo(p)$.

Thm $f \in C^2 \Rightarrow f \in D_A$. and satisfies:

$$Af = \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum (\sigma\sigma^T)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Pf: 1°) x is trap for (X_t).

Then $Af(x) = 0$.

choose V open, bdd. $V \ni x$. Modify f to f_0 outside V. s.t. $f_0 \in C_c^2(\mathbb{R}^2)$.

Note: $\mathbb{E}^x(f_0(X_t)) = f(x) \Rightarrow Af(x) = 0$.

2) x is not a trap for (X_t).

$\exists U$. nbd of x. s.t. $\mathbb{E}^x(\tau_U) < \infty$.

WLOG. consider $U_k \subset U$. $\tau_k =: \tau_{U_k}$.

$$\left| \frac{\mathbb{E}^x(f(X_{\tau_k})) - f(x)}{\mathbb{E}^x(\tau_k)} - Af(x) \right| =$$

$$\left| \mathbb{E}^x\left(\int_0^{\tau_k} Af(X_s) - Af(x) \, ds \right) \right| / \mathbb{E}^x(\tau_k)$$

$$\le \sup_{y \in U_k} |Af(y) - Af(x)| \xrightarrow{k \to \infty} 0$$

since $Af \in C(\mathbb{R}^n)$.

RMK: So we prove: $D(A) \subset D(\mathcal{A}) \supset C^2$. So:

An Itô diffusion is a diffusion in sense of Dynkin's. (Defined by conti. (X_t) with A generator. $\forall f \in C_0^2$. st.

$\mathbb{E}^x(f(X_t)) = f(x) + \mathbb{E}^x\left(\int_0^t Af(X_s) ds\right)$)

(3) Kolmogorov's Backward Equation:

Thm. For $f \in C_c^2(\mathbb{R}^n)$. Define: $u(t,x) = \mathbb{E}^x(f(X_t))$

Then $u(t,\cdot) \in D_A$, $\forall t$, and satisfies:

$$\begin{cases} \partial u/\partial t = Au, & t>0,\ x \in \mathbb{R}^n. \\ u(0,x) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

Moreover if $w(t,x) \in C^{1,2}(\mathbb{R}' \times \mathbb{R}^n)$ and satisfies equation above. Then $u = w$. a.s.

Pf: $t \mapsto u(t,x)$ is differentiable. Directly check.

For uniqueness:

If $\widetilde{A}w = Aw - \partial w/\partial t = 0$. $w(0,x) = f(x)$.

Define $Y_t(\omega) = (s-t,\ X_t^{0,x})$. fix (s,x).

which has generator $\widetilde{A}$.

Apply Dynkin's: $\mathbb{E}^{s,x}(w(Y_t)) = w(s,x)$. $\forall t \geq 0$

Rmk: Define semigroup $Q_t : f \mapsto \mathbb{E}^{\cdot}(f(X_t))$

Then resolvent R_α is: $R_\alpha g(x) =$

$\int_0^\infty e^{-\alpha t} Q_t g(x) \overset{\text{Fubini}}{=} \mathbb{E}^x(\int_0^\infty e^{-\alpha t} g(X_t)dt)$

for $\alpha > 0$. $g \in C_B$. (since $\|Q_t\| \leq 1$).

Lemma. $g \geq 0$. measurable on $\mathbb{R}^n$. Def: $u(x) = Q_t g(x)$.

i) g is l.s.c $\Rightarrow$ u is l.s.c

ii) $g \in C_B$ $\Rightarrow$ u is conti.

Pf: By Pf of uniqueness part of E and U Thm:

$\mathbb{E}(|X_t^x - X_t^y|^2) \le C(t)|x-y|^2$. (From Gronwall)

So: We can find seq $(x_n) \to x$.

St. $X_t^{x_n} \to X_t^x$ a.s.

i) $u(x) = \mathbb{E}(g(X_t^x)) \overset{\text{l.s.c}}{\le} \mathbb{E}(\underline{\lim}_n g(X_t^{x_n}))$

$\overset{\text{Fatou's}}{\le} \underline{\lim}\, \mathbb{E}(g(X_t^{x_n})) = \underline{\lim}\, u(x_n)$

ii) Apply i) on g and $-g$.

(4) Feynman - Kac Formula:

Thm. $f \in C_c^2(\mathbb{R}^n)$. $q \in C(\mathbb{R}^n)$. $q \ge c$. A is generator of (X_t).

If $v(t,x) = \mathbb{E}^x(e^{-\int_0^t q(X_s)ds} f(X_t))$ Then:

$$\begin{cases} \frac{\partial v}{\partial t} = Av - qv & t>0.\ x \in \mathbb{R}^n. \\ v(0,x) = f(x) & x \in \mathbb{R}^n. \end{cases}$$

Moreover, if $w(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$. and bdd on every $K \times \mathbb{R}^n$. K is cpt. satisfies the equation above. then $w = v$.

Pf: Set $Y_t = f(X_t)$. $Z_t = e^{-\int_0^t q(X_s)ds}$

$dZ_t = -Z_t q(X_t)dt$. $dY_t = \cdots$ (Itô).

$d(Y_t Z_t) = Y_t dZ_t + Z_t dY_t$. ($dZ_t dY_t = 0$)

$\Rightarrow Y_t Z_t$ is Itô process.

So $V(t,x) = \mathbb{E}^x(Y_t Z_t)$ is differentiable at t.

$\frac{1}{r}(\mathbb{E}^x(V(t,X_r)) - V(t,x))$

$= \frac{1}{r}(\mathbb{E}^x(\mathbb{E}^{X_r}(Z_t f(X_t))) - \mathbb{E}^x(Z_t f(X_t))$

$\overset{\text{markov}}{=} \frac{1}{r}(\mathbb{E}^x(\mathbb{E}^x(f(X_{t+r}) e^{-\int_0^t q(X_{s+r})ds} | \mathcal{F}_r) - \mathbb{E}^x(Z_t f(X_{t+r}))$

$= \frac{1}{r}\mathbb{E}^x(Z_{t+r} f(X_{t+r}) e^{\int_0^r q(X_s)ds} - Z_t f(X_t))$

$= \frac{1}{r}\mathbb{E}^x(f(X_{t+r}) Z_{t+r} - f(X_t) Z_t) + \frac{1}{r}\mathbb{E}^x(\cdots)$

$\to \frac{\partial v}{\partial t} + q(x) V(t,x)$.

For uniqueness:

Set: $H_t = (s-t, X_t^{0,x}, Z_t)$, $Z_t = z + \int_0^t q(X_s)ds$.

fix $(s,x,z) \in \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^1$

If w satisfies: $\hat{A}w =: -\frac{\partial w}{\partial t} + Aw - qw = 0$

Note H has generator $A_H = -\frac{\partial}{\partial s} + A + q(x)\frac{\partial}{\partial z}$

Argue as before:

Set $\phi(s,x,z) = w(s,x)e^{-z}$ (bdd). $A_H\phi = 0$.

Apply Dynkin's on $\phi(H_{t\wedge\tau_R})$. set $R \to \infty$.

<u>Rmk:</u> (Kill ~ Diffusion)

Note for $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.

(X_t) has generator A:

$$Af(x) = \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}$$

It's natural to ask:

If we can find processes whose generator has form: $\tilde{A}f = \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} - cf$.

where $c(x) \in C_B(\mathbb{R}^n)$.

Def: A killing time kills X_t is a random time ζ. s.t. if set $\tilde{X}_t = X_t$ for $t < \zeta$. $\tilde{X}_t = \partial \notin \mathbb{R}^n$. a coffin state. $\Rightarrow$ $(\tilde{X}_t)$ is strong Markov process and $\mathbb{E}^x(f(\tilde{X}_t)) =: \mathbb{E}^x(f(X_t)\chi_{(0,\zeta)}) = \mathbb{E}^x(f(X_t)e^{-\int_0^t c(X_s)ds})$, for $\forall f \in C_B(\mathbb{R}^n)$ for some $c(s) \geq 0$

Rmk: i) If $c \geq 0$. Then killing time ζ associated with $c(x)$. always exists.

So: $(\tilde{X}_t)$ is the process with generator $\tilde{A} = A - c(x)$. by FK formula.

ii) $c(x)$ is interpreted as killing rate:

$c(x) = \lim_{t \downarrow 0} \frac{1}{t} Q^x(X_0$ is killed in time interval $(0,t])$ ("killed" = "in coffin")

iii) If $c \geq 0$. Then ζ can be constructed explicitly.

(5) Mart. Problem:

① Thm. If X_t is Itô diffusion with generator A. Then $\forall f \in C_c^2(\mathbb{R}^n)$. $M_t = f(X_t) - \int_0^t Af(X_s)ds$ is a mart. w.r.t $(\mathcal{M}_t)$.

Pf: If $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.

Then: $M_t = f(x) + \int_0^t \nabla f^T(X_r)\sigma(X_r)dB_r$.

by Itô formula. (Itô integral is mart.)

Def: L is semi-elliptic differential operator of form: $L = \sum b_i \frac{\partial}{\partial x_i} + \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$, b_i, a_{ij} are locally bdd Borel on $\mathbb{R}^n$.

$\tilde{P}^x$ on $(C(\mathbb{R}^n)^{[0,\infty)}, \mathcal{B})$ solves mart problem for L (start at x) if process

$$\begin{cases} M_t = f(X_t^x) - \int_0^t Lf(X_r^x)dr & \text{is } \tilde{P}^x\text{-mart} \\ M_0 = f(x) \end{cases}$$

w.r.t $\mathcal{B}_{(\mathbb{R}^n)^{[0,t]}}$ for $\forall f \in C_c^2(\mathbb{R}^n)$.

where $X_t^x(w) = W_t$, $W \in \Omega$. canonical process.

The mart problem is well-posed if $\tilde{P}^x$ is unique p.m. to solve it.

Thm If $\tilde{Q}^x$ is p.m on $(\Omega, \mathcal{M}) = (C(\mathbb{R}^n)^{[0,\infty)}, \mathcal{B})$ induced by the law Q^x of Itô diffusion $(X_t)_{t\geq 0}$ with generator A. Then $\tilde{Q}^x$ solves mart. problem for operator A.

Rmk: It also holds for $(X_t)_t$ is even the weak solution of $dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt$

<u>Cor.</u> If $\tilde{P}^x$ solves the mart. problem for

$L = \sum b_i \frac{\partial}{\partial x_i} + \sum \frac{1}{2}(\sigma\sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$. Then:

there exists a weak solution (X_t) of

$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$.

Moreover, the mart. problem for L is

well-posed $\Leftrightarrow$ $(X_t)_{t\geq 0}$ is Markov process.

<u>Rmk:</u> If b, σ satisfies Lipschitz condition

Then, $\tilde{P}^x$ induced by Law of $X_t = x_t$

is the unique solution for mart problem.

(But it's not necessary condition)

<u>Thm.</u> (Stroock. Varadhan)

$L = \sum b_i \frac{\partial}{\partial x_i} + \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ has unique solution

for mart. problem if (a_{ij}) is positive definite. $a_{ij}(x)$ is conti. $b(x)$ measurable. and $\exists D$.

s.t. $|b(x)| + \|a(x)\|^{\frac{1}{2}} \leq D(1+|x|)$.

② Note that $(X_t)_{t\geq 0}$ is Itô process $\Rightarrow$ $(\phi(X_t))_{t\geq 0}$

is Itô process as well. for $\phi \in C^2(\mathbb{R}^n)$

Next. we will find conditions s.t. (X_t) is Itô diffusion

$\Rightarrow$ $\phi(X_t)$ is Itô diffusion as well.

Thm. $dX_t = b(X_t)dt + \sigma(X_t)dB_t$. $b \in \mathbb{R}^n$. $\sigma \in \mathbb{R}^{n\times m}$. $X_0 = x$.

$dY_t = u(t,w)dt + v(t,w)dB_t$. $u \in \mathbb{R}^n$. $v \in \mathbb{R}^{n\times m}$. $Y_0 = x$.

Then $(X_t) \overset{d}{\sim} (Y_t)$. $\Longleftrightarrow$ $vv^T(t,w) = \sigma\sigma^T(Y_t^x)$.

and $\mathbb{E}^x(u(t,\cdot) \mid N_t) = b(Y_t^x)$. $dt \times dP$ a.s.

where $N_t = \sigma(Y_s, s \le t)$.

Pf: ($\Leftarrow$) Suppose $A = \sum b_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum(\sigma\sigma^T)_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$

is generator of (X_t).

Def: $Hf(t,w) = \sum u_i(t,w)\frac{\partial f}{\partial x_i}(Y_t) + \frac{1}{2}\cdot$

$\sum (vv^T)_{ij}(t,w)\frac{\partial^2 f}{\partial x_i \partial x_j}(Y_t)$ for $f \in C_c^2$.

Note: $\mathbb{E}^x(f(Y_s) \mid N_t) =$ (Itô Formula)

$f(Y_t) + \mathbb{E}^x(\int_t^s Hf(r,w)dr \mid N_t)$

$= f(Y_t) + \mathbb{E}^x(\int_t^s \mathbb{E}(Hf(r,w) \mid N_r) \mid N_t)$

$\overset{\text{cond.}}{=} f(Y_t) + \mathbb{E}^x(\int_t^s Af(Y_r)dr \mid N_t)$

Where $\mathbb{E}^x$ is expectation under law of (Y_t)

$\Rightarrow M_t = f(Y_t) - \int_0^t Af(Y_s)ds$ is mart.

w.r.t. R^x. law of $(Y_t)_{t\ge 0}$

Since (X_t) is Markov $\Rightarrow$ By uniqueness:

$(X_t) \overset{d}{\sim} (Y_t)$. i.e. $R^x = Q^x$.

($\Rightarrow$) $\lim_{h\to 0}\frac{1}{h}(\mathbb{E}^x(f(Y_{t+h}) \mid N_t) - f(Y_t)) \overset{(\text{Itô})}{=}$

$\sum \mathbb{E}^x(u_i(t,w) \mid N_t)\frac{\partial f}{\partial x_i}(Y_t) + \frac{1}{2}\sum \mathbb{E}^x((vv^T)_{ij} \mid N_t)\frac{\partial^2 f}{\partial x_i \partial x_j}(Y_t)$

Besides, since $X_t \sim Y_t$, so Y_t is Markov.

$$LHS = \lim_{h \downarrow 0} \frac{1}{h} (\mathbb{E}^{Y_t} (f(Y_h)) - \mathbb{E}^{Y_t} (f(Y_0)))$$

$$= \sum \mathbb{E}^{Y_0} (u_i(0,\omega)) \frac{\partial f}{\partial x_i} (Y_t) + \frac{1}{2} \mathbb{E}^{Y_t} ((vv^T)(0)) \frac{\partial^2 f}{\partial x_i \partial x_j} (Y_t)$$

$$\Rightarrow \begin{cases} \mathbb{E}^x (u(t,\omega) \mid N_t) = \mathbb{E}^{Y_t} (u(0,\omega)) = b(Y_t) \\ \mathbb{E}^x (vv^T(t,\omega) \mid N_t) = \mathbb{E}^{Y_t} (vv^T(0,\omega)) = \sigma\sigma^T(Y_t) \end{cases}$$

The conclusion follows from the next Lemma:

<u>Lemma</u>. There exists an N_t-adapted process $W(t,\omega)$ s.t. $vv^T(t,\omega) = W(t,\omega)$ a.s.

<u>Pf</u>: It's directly from Itô formula:

$$Y_i Y_j (t,\omega) = x_i x_j + \int_0^t Y_i \, dY_j + \int_0^t Y_j \, dY_i + \int_0^t (vv^T)_{ij} (s,\omega) \, ds.$$

<u>Rmk</u>: $u(t,\cdot)$ and $\sigma(t,\cdot)$ may not be N_t-adapted.

<u>Cor</u>. (Recognize a Bm)

A Itô process $dY_t = u(t,\omega) dt + v(t,\omega) dB_t$ is Bm $\Leftrightarrow \mathbb{E}^x (u(t,\cdot) \mid N_t) = 0$, $vv^T(t,\omega) = I_n$.

Thm. $\phi(X_t)$ image of Itô diffusion X_t by C^2 func.

$\overset{d}{\sim} Z_t$ (Itô diffusion) $\Leftrightarrow A(f \circ \phi) = \hat{A}(f) \circ \phi$

$\forall f = \sum a_i x_i + \sum c_{ij} x_i x_j$ (2^{nd}-order poly's). Where $A, \hat{A}$ are generators of X_t, Z_t.

(6) Random Time Change:

Def: i) For $c(t,\omega) \geq 0$, $\mathcal{F}_t$-adapted. $\beta_t(\omega) = \int_0^t c(s,\omega)ds$. is said to be a random time change with time change rate $c(t,\omega)$.

ii) Right-inverse of β_t is $\alpha_t = \inf\{s \mid \beta_s > t\}$.

Rmk: i) α_t is called right-inverse is from $\beta_t \uparrow$ nondecreasing. So $\alpha(t,\omega)$ is right-conti. Besides, $\beta(\alpha(t,\omega),\omega) = t$.

ii) If $c(t,\omega) > 0$. Then $\beta_t \uparrow$ strictly. $t \mapsto \alpha_t(\omega)$ is conti. So it's also left-inverse of β_t.

iii) α_t is a $\mathcal{F}_s$-stopping time:

$\{\alpha_t < s\} = \{t < \beta_s\} \in \mathcal{F}_s$.

Question: For X_t. Itô diffusion. Y_t Itô process.

$$\begin{cases} dX_t = \sigma(X_t)dB_t \quad b(X_t)dt, & X_0 = x. \\ dY_t = u(t,\omega)dt + v(t,\omega)dB_t & Y_0 = x. \end{cases}$$

When does there $\exists\, \beta_t$. s.t. $Y_{\alpha_t} \overset{d}{\sim} X_t$?

Thm. If $\exists\, c(t,w) \geq 0$. $\mathcal{F}_t$-adapted. s.t. $dt \times dP$-a.s.

$u(t,w) = c(t,w)\, b(Y_t)$. $VV^T(t,w) = c(t,w) \cdot \sigma\sigma^T(Y_t)$.

Then: $Y_{\alpha_t} \overset{\wedge}{\sim} X_t$.

Cor. For $c(t,w) \geq 0$. $Y_t = \int_0^t \sqrt{c(s,w)}\, dB_s$.

where B_s is n-dim BM. Then:

Y_{α_t} is also an n-dim BM.

Rmk: i) If $c(t,w) > 0$. Then $Y_t = \hat{B}_{\beta_t}$. where

$\hat{B}_t$ is n-dim BM (since α_t is inverse)

ii) $c(t,w) \equiv 1$. Then it's special case before.

Next. We want to prove: time change of a Itô integral is again a Itô integral. driven by a different BM. $\tilde{B}_t$:

Lemma. $s \mapsto \alpha(s,w)$ is conti. $\alpha(0,w) = 0$. Fix $t > 0$. s.t.

$\beta_t < \infty$. $\mathbb{E}(\alpha(t)) < \infty$.

Set $t_j = \begin{cases} j/2^k. & \text{if } j/2^k \leq \alpha_t \\ \alpha_t & \text{if } j/2^k > \alpha_t \end{cases}$

Choose r_j. s.t. $\alpha_{r_j} = t_j$. If $f(s,w) \geq 0$. $\mathcal{F}_s$-adapted. bdd. s-conti. Then:

$$\lim_{k\to\infty} \sum f(\alpha_{r_j}, w)\, \Delta B_j \overset{L^2(\Omega,P)}{=} \int_0^{\alpha_t} f(s,w)\, dB_s. \text{ a.s.}$$

where $\Delta B_j = B_{\alpha_{r_{j+1}}} - B_{\alpha_{r_j}}$.

Thm. (Time change Formula for Itô integral)

$c(s,w)$, $\tau(s,w)$ are s-conti. $\tau(0,w)=0$. $\mathbb{E}(\tau_t)<\infty$.

$c(s,w)>0$. For B_s m-dim BM and $v(s,w)\in$

$V_H^{n\times m}$ bdd. s-conti. Def:

$\widetilde{B}_t = \int_0^{\alpha t} \sqrt{c(s,w)}\, dB_s$ as in Lemma above.

Then: $\widetilde{B}_t$ is $\mathcal{F}^{(m)}_{\alpha t}$ - BM and $\int_0^{\tau t} v(s,w)\,dB_s$

$= \int_0^t v(\tau r,w)\sqrt{\alpha_r'(w)}\, d\widetilde{B}_r$. P-a.s.

Rmk: Note: $\tau_r'(w) = (\beta_r^{-1})' = 1/c(\alpha r,w)$.

Pf: $\int_0^{\tau t} v(s,w)\,dB_s \overset{(Lem)}{=} \lim_{k\to\infty} \sum v(\alpha_j,w)\, \Delta B_{\alpha_j}$.

$= \lim_{k\to\infty} \sum v(\alpha_j,w)\sqrt{1/c(\alpha_j,w)}\,\Delta\widetilde{B}_j$

$= \int_0^t v(\alpha r,w)\sqrt{1/c(\alpha r,w)}\, d\widetilde{B}_r$

e.g. (Brownian Motion on unit sphere)

i) $n=2$:

consider $g(t,x) = e^{ix} = (\cos x, \sin x)$.

Set $Y(t) = g(t,B_t) = (\cos B_t, \sin B_t)$

where B_t is 1-dim BM.

$Y(t) = (Y_1(t), Y_2(t))$ called BM on unit cycle.

Satisfies $\begin{cases} dY_1 = -\sin(B_t)\,dB_t - \frac{1}{2}\cos(B_t)\,dt \\ dY_2 = \cos(B_t)\,dB_t - \frac{1}{2}\sin(B_t)\,dt \end{cases}$

i.e. $\begin{cases} dY_1 = -Y_2\, dB_t - \frac{1}{2} Y_1\, dt \\ dY_2 = Y_1\, dB_t - \frac{1}{2} Y_2\, dt. \end{cases}$

ii) $n \geq 3$:

Consider $\phi(x) = x/|x|$. $x \in \mathbb{R}^n / \{0\}$.

Set $Y_t = (Y_1(t) \cdots Y_n(t)) = \phi(\vec{B}_t)$. $\vec{B}_t$ is n-dim BM.

By Itô : $dY = \frac{1}{|B|} \cdot \sigma(Y)\, dB + \frac{1}{|B|^2} b(Y)\, dt$.

where $\sigma_{ij}(Y) = \delta_{ij} - Y_i Y_j$. $b(y) = -\frac{n-1}{2} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

Note $c(s,w) = 1/|B_s(w)|^2$. So : $\beta(t,w) = \int_0^t 1/|B_s|^2 ds$.

Set $Z_t(w) = Y_{\alpha(t,w)}(w)$.

By Thm above. $dZ_t = \sigma(Z)\, d\tilde{B} + b(Z)\, dt$. Itô diffusion

We call Z is BM on unit sphere S^{n-1}. Since it's invariant under orthogonal transformation.

Thm.

$\phi : \mathbb{R}^2 \to \mathbb{C}$. $B = (B_1, B_2)$ 2-dim BM. If $\Delta\phi = 0$.

$|\nabla u|^2 = |\nabla v|^2$. $\nabla u \cdot \nabla v = 0$. Then $\phi(B)$ is 2-dim time change BM. in the plane.

Pf: Set $Y = \phi(B) = (u(B_1, B_2), v(B_1, B_2))$

$dY = \frac{1}{2} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} dt + \begin{pmatrix} u_1' & u_2' \\ v_1' & v_2' \end{pmatrix} dB$

So : Y is a mart $\Leftrightarrow$ $\Delta\phi = 0$.

By Cr. above : $\phi(B_1, B_2) = (\tilde{B}^{(1)}_{\beta_1}, \tilde{B}^{(2)}_{\beta_2})$

where $\tilde{B}^{(1)}$. $\tilde{B}^{(2)}$ are two 1-dim BM.

and $\beta_1(t,\omega) = \int_0^t |\nabla u|^2 (B_1, B_2)\, ds$

$\beta_2(t,\omega) = \int_0^t |\nabla v|^2 (B_1, B_2)\, ds$.

If $|\nabla u|^2 = |\nabla v|^2$. $\nabla u \cdot \nabla v = 0$. Then:

$\sigma\sigma^T = |\nabla u|^2 (B_1, B_2) I \Rightarrow Y = \vec{B}_{\beta_t}$.

Rmk: Lévy have proved:

$\phi(B_1, B_2)$ is time change 2-dim

BM $\Leftrightarrow \Delta \phi = 0$.

(7) Girsanov Thm for Diffusion:

It claims: i) If we change drift coefficient of a given Itô process. Then: the law won't change dramatically

ii) Moreover. the law of new process will be absolutely conti. w.r.t the original process.

Lemma: (Bayes' Rule)

For p.m. μ, ν on $(\Omega, \mathcal{G})$. st. $d\nu(\omega) = f(\omega) d\mu(\omega)$ for some $f \in L^1$. If X is r.v. st. $E_\nu(|X|) = \int |X| f(\omega) d\mu < \infty$.

Then $E_\nu(X|\mathcal{H}) E_\mu(f|\mathcal{H}) = E_\mu(fX|\mathcal{H})$ for $\forall \mathcal{H} \subset \mathcal{G}$. sub σ-algebra.

Pf: 1) $\forall H \in \mathcal{H}$. $\int_H \mathbb{E}_\nu(X|\mathcal{H}) f\, d\mu = \int_H \mathbb{E}_\mu(fX|\mathcal{H})\, d\mu$

2) On the other hand:

$$LHS = \mathbb{E}_\mu(\mathbb{E}_\nu(X|\mathcal{H}) f \cdot \chi_H)$$

$$= \mathbb{E}_\mu(\mathbb{E}_\mu(\mathbb{E}_\nu(X|\mathcal{H}) f|\mathcal{H}) \chi_H)$$

$$= \mathbb{E}_\mu(\mathbb{E}_\nu(X|\mathcal{H})\, \mathbb{E}_\mu(f|\mathcal{H}) \chi_H)$$

Thm. (Girsanov Thm. Version I)

$Y(t)$ is Itô process: $dY_t(\omega) = a(t,\omega)dt + dB_t$.

and $Y_0 = 0$. for T (fix) $\leq \infty$. B_t is n-dim BM.

If $M_t = e^{-\int_0^t a(s,\omega)dB_s - \frac{1}{2}\int_0^t a^2(s,\omega)ds}$ is P-mart.

w.r.t $(\mathcal{F}_t^{(n)})_{t\leq T}$ Def: $dQ(\omega) = M_T(\omega)\, dP(\omega)$.

Then: Q is p.m. on $\mathcal{F}_T^{(n)}$. s.t. Y_t is n-dim BM. w.r.t. Q. $0 \leq t \leq T$.

Rmk: i) $P \to Q$ is called Girsanov trans. of measure.

ii) Recall Novikov condition: $\mathbb{E}_P(e^{\frac{1}{2}\int_0^T a^2 ds}) < \infty$ guarantee $(M_t)_{t\leq T}$ is P-mart.

iii) M_t is P-mart $\Rightarrow M_T\, dP = M_t\, dP$ on $\mathcal{F}_t^{(n)}$. $\forall\, 0 \leq t \leq T$.

Pf: Check $\mathbb{E}_P(f M_T) = \mathbb{E}_P(f M_t)$.

$\forall f \in \mathcal{F}_t$. bdd.

Pf: 1) $Q(\Omega) = \mathbb{E}_Q(1) = \mathbb{E}_P(M_T) = \mathbb{E}_P(M_0) = 1$.

So Q is a p.m.

2) Next, check Lévy charac. on $\vec{Y}_t$:

First, prove $Y_i(t)$ is Q-mart. $\forall\, 1 \leq i \leq n$.

Note set $R_t = -\int_0^t a(s,w)\,dB_s - \frac{1}{2}\int_0^t a^2\,ds$

$$dM_t = d e^{R_t} = e^{R_t}\,dR_t + \frac{e^{R_t}}{2}(dR_t)^2$$

$$= M_t \sum_1^n (-a_i(t)\,dB_i(t))$$

$\Rightarrow d(Y_i(t)M(t)) = M_t\, Y^{(i)}(t)\,dB_t$. P-mart.

$$Y_j^{(i)} = \begin{cases} -Y_i(t)\,a_j(t) & j \neq i \\ 1 - Y_i(t)\,a_i(t) & j = i \end{cases}$$

By Bayes's Lemma:

$$\mathbb{E}_Q(Y_i(t) \mid \mathcal{F}_s) = \frac{\mathbb{E}_P(M_t Y_i \mid \mathcal{F}_s)}{\mathbb{E}_P(M_t \mid \mathcal{F}_s)} = Y_i(s)$$

Analogously, $(Y_i(t)Y_j(t) - \delta_{ij}t)M_t$ is P-mart.

Rmk: i) Note $M_T > 0$. a.s. So $Q \sim P$.

ii) Note we shift (B_t) by $a(t,w)$. Then: it's still BM under Q if some mart. condition holds.

Thm. (Girsanov Thm Version II).

$Y(t) \in \mathbb{R}^n$. $dY(t) = \beta(t,\omega) dt + \theta(t,\omega) dB_t$. $t \le T$.

Itô process. $\beta \in \mathbb{R}^n$. $\theta \in \mathbb{R}^{n\times m}$. B_t is m-dim Bm.

If $\exists\, u(t,\omega) \in W_H^m$. $\alpha(t,\omega) \in W_H^n$. s.t.

$$\theta(t,\omega)\, u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega).$$

and $M_t(\omega) = \exp(-\int_0^t u(s,\omega) dB_s - \frac{1}{2}\int_0^t u^2(s,\omega) ds)$

$t \le T$. is a P-mart.

Then for $dQ = M_T dP$ on $\mathcal{F}_T^{(m)}$. Q is p.m on

$\mathcal{F}_T^{(m)}$. s.t. $\hat{B}(t) =: \int_0^t u(s,\omega) ds + B(t)$. $t \le T$ is

Bm under Q. And $dY_t = \alpha(t,\omega) dt + \theta(t,\omega) d\hat{B}_t$.

Pf: 1') $\hat{B}(t)$ is Bm under Q follows directly

from Version I.

2') The representation is easy to check.

Thm. (Girsanov Thm. Version III. for diffusion)

For X_t^x. $Y_t^x \in \mathbb{R}^n$. Itô diffusion and process.

s.t. $\begin{cases} dX_t = b(X_t) dt + \sigma(X_t) dB_t \\ dY_t = (\gamma(t,\omega) + b(Y_t)) dt + \sigma(Y_t) dB_t. \end{cases}$

b, σ satisfies Lipschitz cond. $\gamma \in W_H^n$.

If $\exists\, u(t,\omega) \in W_H^m$. s.t. $\sigma(Y_t)\, u(t,\omega) = \gamma(t,\omega)$.

suppose M_t. Q. $\hat{B}(t)$ as defined above.

and M_t is P-mart. w.r.t. $\mathcal{F}_t^{(m)}$. Then:

Q is p.m. on $\mathcal{F}_T^{(m)}$. and $dY_t = b(Y_t)dt$

$+ \sigma(Y_t) d\hat{B}_t$. $t \leq T$.

So: Q-law of Y_t^x = P-law of X_t^x.

for $t \leq T$.

Pf: It follows from weak uniqueness.

Rmk: It can be used to produce weak

solution of SDE:

Suppose Y_t is (weak or strong) solution

of $dY_t = b(Y_t)dt + \sigma(Y_t)dB_t$.

We wish to find weak solution for

related SDE: $dX_t = a(X_t)dt + \sigma(X_t)dB_t$.

If $\exists\, u_0 : \mathbb{R}^n \to \mathbb{R}^m$. s.t. $\sigma(\eta)u_0(\eta) = b(\eta) - a(\eta)$.

and satisfies Novikov's condition.

Then: $\exists\, (\hat{B}_t, Q)$. s.t. $(Y_t, \hat{B}_t)$ is its

weak solution under Q.