Stationary Dist.

Thm: For $(X_t)_{t\geq 0}$ Feller process with generator $\mathcal{L}$.

Then for measure μ on E.

i) μ is stationary dist. $\Leftrightarrow$ $\mu(\mathcal{L}f)=0$. $\forall f \in D(\mathcal{L})$. and $\mu(E)=1$.

ii) μ is reversible dist. $\Leftrightarrow$ $\mu(g\mathcal{L}f) = \mu(f\mathcal{L}g)$. $\forall f, g \in D(\mathcal{L})$. $\mu(E)=1$.

Pf: check from: $\mu(P_t f) = \mu(f)$.

$\mu(g P_t f) = \mu(f P_t g)$.

Denote: For Itô Diffusion: $dX_t = b(X_t)dt + \sigma(X_t) dB_t$. Set: $s(x) = \int_{x_0}^{x} e^{-\int_{x_0}^{y} 2b(u)/\sigma^2(u)\, du} dy$.

$$m(x) = \frac{1}{\sigma(x)} e^{\int_{x_0}^{x} 2b(u)/\sigma^2(u)\, du}.$$

Rmk: For generator $\mathcal{L}$ of X_t

$\Rightarrow \mathcal{L}s = 0$.

Lemma: $\tau_m = \inf\{t\geq 0 \mid |X_t| \geq m\}$. $\Rightarrow \tau_m \xrightarrow{m\to\infty} \infty$ a.s.

Pf: Note $X_{t\wedge\tau}^2 - X_0^2 - \int_0^{t\wedge\tau} (\sigma^2(X_s) + 2X_s b(X_s))\, ds$ is mart.

By BDG and Lipschitz condition:

$$\mathbb{E}(\sup_{[0,t]} |X_{s\wedge\tau}|^2) \leq x^2 + C_1 t + C_2 \int_0^t \mathbb{E}(\sup_{[0,s]} |X_{u\wedge\tau}|^2) ds$$

$\Rightarrow$ Gronwall's inequi.:

$$\mathbb{E}(\sup_{[0,t]} |X_{s\wedge\tau}|^2) \leq C_1 t e^{C_2 t}.$$

So: $\mathbb{P}(\tau_m \leq T) \leq \mathbb{P}_x(\sup_{[0,T]} |X_t| \geq m)$

$$\leq \mathbb{E}(\sup_{[0,T]} |X_t|^2) / m^2 \to 0$$

Thm. For $x \in [\alpha, \beta]$. $\tau =: \tau_\alpha \wedge \tau_\beta$.

i) $\mathbb{P}_x(\tau_\alpha < \tau_\beta) = (S(x) - S(\alpha)) / (S(\beta) - S(\alpha))$

ii) X_t is recurrent $\Leftrightarrow$ $|S(x)| \xrightarrow{x \to \infty} \infty$

iii) If $Z = \int_{\mathbb{R}'} m < \infty$. Then $\mu(A) = \int_A \frac{m(x)}{Z} dx$

is stationary dist.

Pf: i) Note $S(X_{t\wedge\tau}) - S(X_0)$ is mart. ($\mathcal{L}S = 0$)

ii) By i). Set $x \to \infty$

iii) Check $\mu(\mathcal{L}g) = 0$. $\forall g \in D(\mathcal{L})$.