Boundary Value Problems

(1) Settings:

$D \subset \mathbb{R}^n$. open connected. $L = \sum b_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j} a_{ij}(x) \cdot \frac{\partial^2}{\partial x_i \partial x_j}$ semi-elliptic linear diff. operator on $C^2(\mathbb{R}^n)$. where b_i. $a_{ij} = a_{ji}$ are conti. and $a(x) =: (a_{ij}(x))_{n\times n} \geq 0$.

Consider: $\phi \in C(\partial D)$. $g \in C(D)$. given.

Find $w \in C^2(D)$. s.t.
$$\begin{cases} Lw = -g & \text{in } D \\ \lim\limits_{\substack{x \to y \\ \in D}} w(x) = \phi(y) & \forall y \in \partial D. \end{cases}$$

Idea: Find Itô diffusion (X_t) with generator $A = L$ on $C_0^2(\mathbb{R}^n)$. Set $\sigma(x)\sigma^T(x) = 2a(x)$.

$\Rightarrow dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$.

Thm. (uniqueness)

If ϕ is bdd and $\mathbb{E}(\int_0^{\tau_D} |g(X_t)|) < \infty$. $w \in C_B^2(D)$ satisfies: $Lw = -g$ on D. and $\lim\limits_{t \uparrow \tau_D} w(X_t) = \phi(X_{\tau_D}) I_{\tau_D < \infty}$. Then:

$w(x) = \mathbb{E}^x(\phi(X_{\tau_D}) I_{\tau_D < \infty}) + \mathbb{E}^x(\int_0^{\tau_D} g(X_t))$

Pf: By Dynkin's formula. Approxi. τ_D by bdd stopping times.

(2) Dirichlet's Problem:

$\phi \in C(\partial D)$. given. Find $u \in C^2(D)$. st.

$$\begin{cases} Lu = 0. & \text{in } D \\ \lim\limits_{\substack{x \to y \\ \in D}} u(x) = \phi(y) & \forall y \in \partial D. \end{cases}$$

Rmk: We may think $u(x) = \mathbb{E}^x(\phi(X_{\tau_D}))$ is a plausible solution. But it won't even be conti. naturally.

① Stochastic Dirichlet Prob:

Denote A is characteristic opera. of X_t.

Def: f is locally bdd. measurable on D. it's X-harmonic in D if:

$f(x) = \mathbb{E}^x(f(X_{\tau_U}))$ $\forall x \in D$. $\bar{U} \subset D$. open bdd.

Rmk: It's equi. : supermeanvalued f. $\forall \tau_k \to 0$. a.s. $f(x) = \lim\limits_k \mathbb{E}^x(f(X_{\tau_k}))$

Lemma f is X_t-harmonic in D $\underset{f \in C^2(D)}{\rightleftarrows}$ $Af = 0$ in D.

Pf: By Dynkin's formula.

Note: $D(L) \subset D(A)$. $L = A$ on $D(L)$.

Lemma φ is bdd. measurable on ∂D. Then:

$u(x) = \mathbb{E}^x(\varphi(X_{\tau_D}))$ is X_t-harmonic.

Pf: By mean value prop. of diffusion.

Thm. For bdd measurable φ on ∂D. Sto-Diri. Problem is find u on D. s.t.

$$\begin{cases} u \text{ is } X_t\text{-harmonic} \\ \lim\limits_{t \uparrow \tau_D} u(X_t) = \varphi(X_{\tau_D}) \text{ a.s. } x \in D. \end{cases}$$

i) (Existence)

$u(x)$ above solves SDP.

ii) (Uniqueness)

$\forall\, g$ bdd on D solves SDP must equal $u(x)$.

Pf: ii) By BDT.

i) Only need to check boundary cond.

Fix $x \in D$. D_k open $\uparrow D$. $D_k \subset\subset D$.

$\tau_k =: \tau_{D_k}$. $\tau =: \tau_D$. $\tau_k \uparrow \tau_D$

$M_k = u(X_{\tau_k}) = \mathbb{E}^x(\theta_{\tau_k}\varphi(X_\tau) \mid \mathcal{F}_{\tau_k})$

$= \mathbb{E}^x(\varphi(X_\tau) \mid \mathcal{F}_{\tau_k})$. bdd mart.

$\Rightarrow$ It converges to $\varphi(X_\tau)$. in L^p. $\forall p \geq 1$

By Doob's inequi: $\mathbb{P}^x(\sup |u(X_t) - \cdots| > \varepsilon) \leq \cdots$

$\Rightarrow$ It converges. a.s.

Rmk: There's a distance between solution of SDP and generalized Dirichlet problem.

② **Generalized Dirichlet prob:**

Lemma. (Blumenthal's law for diffusion)

Q^x is law of diffusion X_t.

$H \in \bigcap_{t>0} \mathcal{F}_t \Rightarrow Q^x(H) \in \{0,1\}$.

Pf: By Markov prop. for $\forall \eta : \Omega \to \mathbb{R}^1$ bdd. measurable: $E^x(\theta_t \eta \mid \mathcal{F}_t) = E^{X_t}(\eta)$.

So: $\int_H \theta_t \eta \, dQ^x = \int_H E^{X_t}(\eta) \, dQ^x$. $\forall t$.

replace η by $\prod_1^n g_i$. $g_i \in C_B$.

Let $t \to 0$. $\eta = I_H$. So: $Q^x(H) = Q^x(H)^2$.

Rmk: It also holds for general Feller process.

Cor. $\mathbb{P}^x(\tau_D = 0) \in \{0,1\}$. $\forall x \in D$.

Def: i) $y \in \partial D$ is regular for D. w.r.t X_t if $Q^y(\tau_D = 0) = 1$.

Rmk: All the boundary points will stay on $\bar{D}$ or leave immediately.

ii) measurable set $A \subset \mathbb{R}^n$ is thin for X_t

if $Q^x(T_A = 0) = 0.$ $\forall x.$ $T_A =: \{t > 0 \mid X_t \in A\}.$

iii) Semipolar set is countable union of thin sets.

iv) Hunt's condition: Any semipolar set for

X_t is also polar for X_t.

Rmk: BM indeed holds Hunt's cond.

Cor. Itô diffusion with $b(x)$ satisfy

Novikov cond. and $a(x)$ has

bdd inverse also holds Hunt's

condition.

Pf: By Girsanov represent.

Lemma $U \subset D$ open and I is set of irreg.

points of U. $\Rightarrow$ I is semipolar.

Rmk: It's intuitive that countable

operation can retain "thin".

Thm. (Uniqueness)

For X_t satisfies Hunt's condition and

$\phi \in C_B(\partial D)$. If $u \in C^2(D)$ satisfies:

$$\begin{cases} Lu = 0 & \text{in } D \\ \lim\limits_{\substack{x \to y \\ D}} u(x) = \phi(y). & \forall y \in \partial D. \text{ regular} \end{cases} \quad \text{Then: } u(x) = \mathbb{E}^x(\phi(X_{\tau_D}))$$

Pf: As argument before. By Lemma and Hunt's cond. $\Rightarrow$ X_{τ_D} & I. a.s.

So: $u(x) = \lim_k \mathbb{E}^x(u(X_{\tau_k})) = \mathbb{E}^x(\phi(X_{\tau_D}))$

Thm. If L is uniformly elliptic in D. $\phi \in C_b(\partial D)$, $u(x) =: \mathbb{E}^x(\phi(X_{\tau_D}))$. Then:

$u \in C^{2+\alpha}(D)$. $\forall \alpha < 1$. and solves the problem in the Uniqueness Thm. above.

(3) Poisson Problem:

Thm. (Stochastic Poisson problem)

For $g \in C(D)$. If $\mathbb{E}^x(\int_0^{\tau_D} |g(X_t)|) < \infty$. $\forall x \in D$.

$v(x) =: \mathbb{E}^x(\int_0^{\tau_D} g(X_t))$. Then: v solves

$$\begin{cases} Av = -g & \text{in } D \\ \lim\limits_{\substack{x \to y \\ x \in D}} v(x) = 0. & \forall \text{ regular } y \in \partial D. \end{cases}$$

Pf: 1) $\forall x \in D$. find U open. $x \in U \subset\subset D$. $\tau =: \tau_U$.

Set $\eta = \int_0^{\tau_D} g(X_t)\,dt$.

So: $\dfrac{\mathbb{E}^x(v(X_\tau)) - v(x)}{\mathbb{E}^x(\tau)} \overset{\text{MP.}}{=} \dfrac{\mathbb{E}^x(\theta_\tau \eta - \eta)}{\mathbb{E}(\tau)}$

$= - \dfrac{\mathbb{E}^x(\int_0^\tau g(X_s))}{\mathbb{E}(\tau)} \xrightarrow[U \downarrow x]{\tau \to 0} -g(x).$

ii) By DCT. directly.

<u>Thm</u> (Uniqueness for generalized Poisson problem)

For X_t satisfies Hunt's condition and $\mathbb{E}^x(\int_0^{\tau_D} |g(X_t)|) < \infty$. $g \in C(D)$. given. If

$\exists\, v \in C^2(D)$ satisfies: $|v(x)| \leq C(1+\mathbb{E}^x(\int_0^{\tau_D} |g(X_t)|))$

$\forall x \in D$. and $\begin{cases} Lv = -g. & \text{in } D. \\ \lim\limits_{\substack{x \to y \\ x \in D}} v(x) = 0 & \forall y \text{ regular in } \partial D \end{cases}$

Then $v(x) = \mathbb{E}^x(\int_0^{\tau_D} g(X_t)\,dt)$.

<u>Pf:</u> Bound τ_D by (τ_k).

Using Dynkin's again and DCT.