Application on Stochastic Control.

(1) <u>Setting</u> :

Consider state of system at time t :

$dX_t = b(t, X_t, u_t)\,dt + \sigma(t, X_t, u_t)\,dB_t$.

<u>Rmk</u>: u_t is $\mathcal{F}_t^{(m)}$-adapted. chosen to control the process X_t. $u_t \in \mathcal{U} \underset{\text{Borel}}{\subset} \mathbb{R}^k$.

where $X_t \in \mathbb{R}^n$. $b: \mathbb{R}^{\geq 0} \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$. $\sigma: \mathbb{R}^{\geq 0} \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^{n \times m}$.

Set : $f: \mathbb{R}^1 \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^1$. profit rate func.

$g: \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$. bequest func.

s.t. they're conti.

$G \subset \mathbb{R}^1 \times \mathbb{R}^n$. fixed. $\hat{T} =: \inf\{r > s \mid (r, X_r^{s,x}) \notin G\}$.

Performance func. $J^u(s, x) =: \mathbb{E}^{s,x}\left(\int_s^{\hat{T}} f^{u_r}(r, X_r)\,dr + g(\hat{T}, X_{\hat{T}})\, I_{\{\hat{T} < \infty\}} \right)$

To simplify as usual :

Set $Y_t =: (s+t, X_{s+t}^{s,x})$. $t \geq 0$. Then :

$J^u(y) = \mathbb{E}^y\left(\int_0^{\tau_G} f^{u_t}(Y_t)\,dt + g(Y_{\tau_G})\, I_{\{\tau_G < \infty\}} \right)$.

Next. We want to find control $u^* \in A$.

Some admissible family and $\bar{\Phi}(\eta)$. s.t.

$\bar{\Phi}(\eta) = \sup_{u \in A} J^u(\eta) = J^{u^*}(\eta)$. optimal.

Def: i) When $u(t,\omega) = u(t)$. it's called deterministic control.

ii) When $u(t,\omega)$ is M_t-adapted. $M_t =: \sigma(X_r^u.\ r \leq t)$. it's called feedback control.

iii) When $u(t,\omega) = u(t, X_t(\omega))$. it's called Markov control since it's Markov process.

(2) HJB equations:

Thm. Def: $\bar{\Phi}(\eta) =: \sup\{ J^u(\eta) \mid u = u(Y).$ Markov control$\}$.

If $\bar{\Phi} \in C^2(G) \cap C(\bar{G})$. $E^\eta(|\bar{\Phi}(Y_\tau)| + \int_0^\tau |L^v \phi(Y_t)| dt) < \infty$. for $\forall$ bdd stopping time $\tau \leq \tau_G$. and L^v is generator of Y_t^v. Besides, the optimal control u^* exists, and ∂G is regular. Then:

i) $\sup_{v \in U} \{ f^v(\eta) + (L^v \bar{\Phi})(\eta) \} = 0$. $\forall \eta \in G$ and

$\bar{\Phi}(\eta) = g(\eta)$. $\forall \eta \in \partial G$.

ii) $f^{u^*(\eta)}(\eta) + (L^{u^*(\eta)} \bar{\Phi})(\eta) = 0$. obtain supremum.

<u>Pf:</u> i) The second assert is from Dirichlet prob.

To prove the first one:

1') Check: $\mathbb{E}^\eta(J^u(Y_\alpha)) = J^u(\eta) - \mathbb{E}^\eta(\int_0^\alpha f(Y_r))$

by SMP. $\forall \eta \in G$. $\alpha \le \tau_G$.

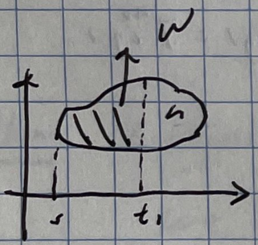
2') Set $W = \{(s,z) \in G \mid s < t_1\}$. $\alpha = \tau_W$.

$$u(r,z) = \begin{cases} v. & (r,z) \in W. \\ u^*. & (r,z) \notin W. \end{cases}$$

So: $\Phi(Y_\alpha) = J^{u^*}(Y_\alpha) = J^u(Y_\alpha)$.

Note: $\Phi(\eta) \overset{(A)}{\geq} J^u(\eta)$. $\forall u$. M.control.

$= \mathbb{E}^\eta(J^u(Y_\alpha)) + \mathbb{E}^\eta(\int_0^\alpha f^v(Y_r))$



Apply Dynkin's on $\mathbb{E}^\eta(\Phi(Y_\alpha))$

$\Rightarrow \mathbb{E}^\eta(\int_0^\alpha f^v(Y_r) + \Phi(Y_r)\,dr) / \mathbb{E}(\alpha) \le 0$.

Set $\alpha \to 0$ (i.e. $t_1 \to s$).

3') Note when $u = u^*$. (A) will be "$=$".

ii) By Poisson problem.

<u>Rmk:</u> It claim: If u^* exists. Then. it's necessary to be optimal for the function $v \mapsto f^v(\eta) + (L^v\Phi)(\eta)$. at $v = u^*(\eta)$. $\forall \eta \in G$.

<u>Thm.</u> (Sufficient)

For $\Phi \in C^2(G) \cap C(\bar{G})$. If:

i) $f^v(y) + (L^v \Phi)(y) \leq 0$. $\forall y \in G$. $v \in U$.

ii) $\lim_{t \to \tau_G} \Phi(Y_t) = g(Y_{\tau_G}) I_{\{\tau_G < \infty\}}$.

iii) $(\Phi(Y_\tau))_{\tau \leq \tau_G}$ is u.i. Then:

i) $\Phi(y) \geq J^u(y)$. $\forall$ Markov control u.

ii) If $\forall y \in G$. $\exists u_0(y)$. St.

$f^{u_0(y)}(y) + L^{u_0(y)} \Phi(y) = 0$. and.

$(\Phi(Y_\tau^{u_0(y)}))_{\tau \leq \tau_G}$ is u.i. Then:

u_0 is the Markov control. St. $\Phi(y) = J^{u_0}(y)$.

<u>Rmk:</u> u_0 won't always exist:

Only when $b, \sigma, f, g, \partial G$ satisfy some certain conditions.

<u>Pf:</u> By Dynkin's formula on $E^y(\Phi(Y_{\tau_R}))$.

where $\tau_R = R \wedge \tau_G$

and apply Fatou's Lemma

<u>Thm.</u> (For other controls)

$\Phi_M(y) =: \sup\{J^u(y) \mid u$ is Markov control$\}$.

$\Phi_a(y) =: \sup\{J^u(y) \mid u \in \mathcal{F}_t^{(m)}$ - adapted.$\}$

If $\exists u_0 = u_0(Y)$. optimal Markov control.

St. ∂h is regular w.r.t Y_t^{u}. $\Phi_m \in C_B^2(h) \cap C(\bar{h})$. satisfies:

$\mathbb{E}^{\eta}(|\Phi_m(Y_\tau)| + \int_0^{\tau} |L^u \Phi_m(Y_t)| dt) < \infty$. $\forall \tau \le \tau_h$.

for $\forall$ adapted control u. $\forall y \in h$.

Then : $\Phi_m(\eta) = \Phi_u(\eta)$ on h.

<u>Pf:</u> If u_t is $\mathcal{F}_t^{(m)}$-adapted. $\Rightarrow Y_t$ is Itô.

$$\mathbb{E}(\Phi_m(Y_{T_R})) = \Phi_m(\eta) + \mathbb{E}^{\eta}(\int_0^{T_R} (L^{u_t}\Phi_m(Y_t) dt)$$

$$\le \Phi_m(\eta) - \mathbb{E}^{\eta}(\int_0^{T_R} f(Y_t, u_t) dt)$$

Set $R \to \infty$. So: $\Phi_m(\eta) \ge J^u(\eta)$.

<u>Thm</u> (Minimal case)

set $\varphi(\eta) = \inf_u J^u(\eta)$. $= J^{u^*}(\eta)$. Then all the claims above hold for reversal case.

<u>Pf:</u> $\varphi(\eta) = -\sup_u (-J^u(\eta))$. Then.

we can set $f = -f$. $g = -g$.

(3) <u>Terminal Condition</u>:

Consider $K = \{ u_t$ is Markov control $| \mathbb{E}^{\eta}(M_i(Y_{\tau_h}^u)) = 0$. $1 \le i \le l$. $\}$ where $M = (M_1 \cdots M_l) \in C(\mathbb{R}^{n+1}; \mathbb{R}^l)$ is given function.

We want to find $\bar{\Phi}(\eta) =: \sup_{u \in K} J^u(\eta)$.

<u>Denote:</u> $\bar{\Phi}_\lambda(\eta) =: \sup_u J_\lambda^u(\eta)$. u is Markov control.

where $J_\lambda^u(\eta) =: \mathbb{E}^\eta\left(\int_0^{\tau_D} f(Y_t^u)\,dt + g(Y_{\tau_D}^u)\right.$

$\left. + \vec{\lambda} \cdot M(Y_{\tau_D}^u)\right)$.

<u>Thm.</u> If $\forall \lambda \in \Lambda$, we have $\bar{\Phi}_\lambda$ and u_λ^* solve the optimal problem for $J_\lambda^u(\eta)$. and $\exists \lambda_0 \in \Lambda$. s.t. $\mathbb{E}^\eta(M_i(Y_{\tau_D}^{u^*_{\lambda_0}})) = 0$. $\forall i$.

Then: $\bar{\Phi}(\eta) = \bar{\Phi}_{\lambda_0}(\eta)$. $u^* = u^*_{\lambda_0}$.

<u>Rmk:</u> First find u_λ^* for $\forall \lambda$. Then find λ_0 satisfies boundary cond.

<u>Pf:</u> By def: $J_{\lambda_0}^{u^*_{\lambda_0}} = J^{u^*_{\lambda_0}} \geq J^u$.

for $\forall u \in K$.