Applications in Finance.

Setting: $\mathcal{F}_t^{(m)}$ is σ-algebra w.r.t. m-dim SBM $(B_t)_{t\geq 0}$. $\mathcal{W}^m =: \{f_t \in \mathcal{F}_t^{(m)} \mid \int_s^t f(u,\omega)\,du < \infty \text{ a.s.}\}$.

(1) Definitions:

Def: i) A market $X(t) = (X_0(t), X_1(t), \cdots X_n(t))$ is $\mathcal{F}_t^{(m)}$-adapted Itô process. $0 \leq t \leq T$.

$$\begin{cases} dX_0(t) = \overset{(\rho \text{ is bdd.})}{\rho(t,\omega)} X_0(t)\,dt. \quad X_0(0) = 1. \\ dX_k(t) = \mu_k(\omega,t)\,dt + \sigma_i(\omega)\,dB_t. \quad X_k(0) = x_k. \end{cases}$$

if $X_0(t) \equiv 1$. We call it's normalization

Rmk: i) $X_k(t)$, are prices of asset k.
Note $X_0(t)$ is risk-free since it's lack of diffusion term.

ii) We can normalize the market by setting $\overline{X}_k(t) =: X_k(t) / X_0(t)$.

Note: $X_0(t) = e^{\int_0^t \rho(s,\omega)\,ds} > 0$.

Set $\xi(t) = X_0^{-1}(t) > 0$. Then:

$d\overline{X}_k(t) = \xi(t)\left((\mu_i - \rho X_i)\,dt + \sigma_i\,dB_t\right)$

i.e. $d\overline{X}_t = \xi_t\,(dX_t - \rho_t X_t\,dt)$

ii) A portfolio in market $(X_t)_{t\in[0,T]}$ is (t,w) - measurable and $\mathcal{F}_t^{(m)}$ - adapted. process $\theta(t,w) = (\theta_0(t,w), \cdots \theta_n(t,w))_{t\leq T}$.

<u>Rmk:</u> $\theta(t,w)$ can be seen as the number of unit of assets we hold at t.

iii) Value at time t of portfolio $\theta(t,w)$ is $V(t,w) = \theta(t,w)\cdot X(t,w) = \sum_{i=0}^{n} \theta_i(t) X_i(t)$.

iv) Portfolio $\theta(t)$ is self-financing if:

$$\int_0^T |\theta_0(s) e(s) X(s) + \sum_{1}^{n} \theta_i(s) \mu_i(s)| + \sum_{j=1}^{m} |\sum_{i}^{n} \theta_i(s) r_{ij}(s)|^2 ds$$

$< \infty$. a.s. and $dV(t) = \theta(t)\, dX_t$.

<u>Rmk:</u> i) $\theta(t) \equiv$ const. is self-financing

ii) The first condition is for intergrability.

iii) It stems from the discrete model: $\Delta V(t_k) = V(t_{k+1}) - V(t_k)$ $= \theta(t_k)\,\Delta X(t_k)$. set $\Delta t_k \to 0$

It means no money brought in or taken out. (which only depend on θ_t).

prop. θ is self-financing for X_t.

$\Longleftrightarrow$ θ is self-financing for $\bar{X}_t$.

Pf: $\bar{V}^{\theta}_t =: \theta(t) \cdot \bar{X}(t) = \xi(t) V^{\theta}(t)$.

using Itô: $d\bar{V}^{\theta}(t) = \theta(t)\, d\bar{X}(t)$.

prop. $\theta(t)$ is self-financing $\Longleftrightarrow$ $\theta_0(t) = V^{\theta}(0)$

$+ \xi(t) A(t) + \int_0^t r(s) A(s) \xi(s)\, ds$, where A_t

$= \sum_{i=1}^{n} \left(\int_0^t \theta_i(s)\, dX_i(s) - \theta_i(t) X_i(t) \right)$

Rmk: Given $(\theta_i(s))_{i=1}^{n}$, we can make $\theta(t)$ self-financing by choosing θ_0 as above and choose V_0^{θ} freely.

Pf: $\Longleftrightarrow$ $\sum_{i=0}^{n} \theta_i(t) X_i(t) = V^{\theta}(0) + \sum_{i=0}^{n} \int_0^t \theta_i(s)\, dX_i(s)$.

set $Y_0(t) = \theta_0(t) X_0(t)$.

$\Longleftrightarrow$ $dY_0(t) = r(t) Y_0(t)\, dt + dA(t)$.

Solve it: $\xi(t) Y_0(t) = \theta_0(0) + \int_0^t \xi(s)\, dA(s)$.

Def: A self-financing portfolio $\theta(t)$ is admissible if $\exists\, K = K(\theta) < \infty$ s.t. $V^{\theta}(t) \geq -K$ a.s. for $[0,T] \times \Omega$.

Lemma. $\theta(t)$ is admissible for $X(t)$

$\iff$ $\theta(t)$ is admissible for $\bar{X}(t)$.

Def: An admissible portfolio $\theta(t)$ is arbitrage if $V^{\theta}(0)=0$. $V^{\theta}(t)\geq 0$. a.s. and $\mathbb{P}(V^{\theta}(T)>0.)>0$

Rmk: $\theta(t)$ can generate a profit without risk of losing money.

A market can't exist for a long time if arbitrage exists!

Rmk: These definitions have additional conditions on self-financing portfolio.

Actually. if we only require self-finan on a portfolio. Then we can generate any final value $V^{\theta}(T)$ from it

Def: A measure $\mathbb{Q}$ is equi. (local) mart. measure if $\mathbb{Q}\sim\mathbb{P}$. and $\bar{X}_t$ is (local) mart. w.r.t $\mathbb{Q}$.

Lemma. If equi. local mart. measure $\mathbb{Q}$ exists. Then. market $(X_t)_{t\leq T}$ has no arbitrage.

Pf: $\mathbb{Q} \sim \mathbb{P} \Rightarrow \bar{V}^\theta(t)$ is local mart. w.r.t $\mathbb{Q}$.

with lower bdd $\Rightarrow \bar{V}^\theta(t)$ is supermart.

So: $\mathbb{E}_{\mathbb{Q}}(\bar{V}^\theta(t)) \le \bar{V}^\theta(0) = 0$.

If $\mathbb{P}(\bar{V}^\theta(T) > 0) > 0$. Then $\mathbb{Q}(\bar{V}^\theta(T) > 0) > 0$

which is a contradiction!

Rmk: Actually, the market also satisfies a stronger condition "no free lunch with vanishing risk" (NFLVR)

Thm. i) If exist $u(t,w)$ is (t,w)-measurable and $\mathcal{F}_t^{(m)}$-adapted. $\mathbb{E}(\int_0^T \|u\|^2 dt) < \infty$.

St. $\sigma(t) u(t) = \mu(t) - \rho(t) X(t)$. a.s. (t,w).

and $\mathbb{E}(e^{\frac{1}{2}\int_0^T u^2(t,w) dt}) < \infty$.

Then, the market X_t has no arbitrage.

ii) Conversely. if X_t has no arbitrage.

then $\exists\, u(t,w)$. $\mathcal{F}_t^{(m)}$-adapted. (t,w)-mearable.

satisfies : $\sigma(t,w) u(t,w) \overset{a.s.}{=} \mu(t,w) - \rho(t,w) X(t)$

Pf: i) WLOG. X_t is normalized. So $\rho \equiv 0$.

set $d\mathbb{Q} = e^{-\int_0^T u\, dB_t - \frac{1}{2}\int_0^T u^2 dt} d\mathbb{P}$.

By Girsanov : $\mathbb{Q} \sim \mathbb{P}$. and.

$\tilde{B}_t = \int_0^t u\, ds + B_t$ is $\mathbb{Q}$-BM

$\Rightarrow dX_k(t) = \sigma_k \, d\tilde{B}(t)$. local mart.

ii) Set $F_t = \{ w \in \Omega \mid \sigma u = \mu$ has no solutions$\}$

$= \{ w \in \Omega \mid \exists\, v(t,w)$. s.t. $\sigma^T(t,w) \cdot v(t,w) = 0$. $v(t,w)\mu(t,w) \neq 0 \}$

Def $\theta_i(t,w) = \begin{cases} 0, & w \notin F_t \\ \operatorname{sgn}(v\mu)\, v_i, & w \in F_t \end{cases}$

Choose $V^\theta(0) = 0$, $\theta_0(t)$ st. θ is Self-financing (Note it's also measurable)

Note: $V^\theta(t) = \int_0^T \sum \theta_i \, dX_i(t)$.

$\overset{\text{Def}}{=} \int_0^T I_{F_t} \cdot |v \cdot \mu| \, dt \geq 0$ a.s.

$\Rightarrow I_{F_t} = 0$ a.s. (t,w). $\forall t$.

<u>prop.</u> i) X_t has an arbitrage $\Leftrightarrow$ $\overline{X}_t$ has an arbitrage.

ii) $\overline{X}_t$ has an arbitrage $\Leftrightarrow$ $\exists$ admissible portfolio θ_t. st. $\overline{V}^\theta(0) \leq \overline{V}^\theta(t)$ a.s. and $\mathbb{P}(\overline{V}^\theta(0) < \overline{V}^\theta(T)) > 0$.

<u>Rmk:</u> So for an normalized market, $\overline{V}^\theta(0) = 0$ is not essential.

Pf: i) Note $\bar{V}^\theta(t) = \xi(t) V^\theta(t)$

ii) ($\Leftarrow$) Def $\tilde{\theta}(t)$ by: $\tilde{\theta}_k(t) = \theta_k(t),\ k>0$
choose $\tilde{\theta}_0(t)$

s.t. $\bar{V}^{\tilde{\theta}}(0) = 0$ and satisfies

self-financing.

So: $\bar{V}^{\tilde{\theta}}(t) = \int_0^t \tilde{\theta}(s)\, d\bar{X}(s)$

$= \int_0^t \theta(s)\, d\bar{X}(s)$

$= \bar{V}^\theta(t) - \bar{V}^\theta(0)$

($\Rightarrow$) Choose $\tilde{\theta}_k(t) = \theta_k(t)$. and

$\tilde{\theta}_0(t) = \bar{V}^\theta(0) + \theta_0(t)$ where

θ is an arbitrage for $\bar{X}_t$.

(2) Attainable and Complete:

Lemma. For u is (t,ω)-measurable. $\mathcal{F}_t^{(m)}$-adapted.

and $\mathbb{E}(e^{\frac{1}{2}\int_0^T \|u\|^2 dt}) < \infty$. Set $dQ =$

$\exp(-\int_0^T u\, dB_t - \frac{1}{2}\int_0^T \|u\|^2 dt)\, d\mathbb{P}$ on $\mathcal{F}_T^{(m)}$.

$\tilde{B}(t) = \int_0^t u\, ds + B(t)$. We have:

i) If $F \in L^2(\mathcal{F}_T^{(m)}, Q) \Rightarrow \exists\, \phi$ satisfies

$\mathbb{E}(\int_0^T \phi^2 dt) < \infty$. (t,ω)-measurable and

$\mathcal{F}_t^{(m)}$-adapted. s.t. $F = \mathbb{E}_Q(F) + \int_0^T \phi\, d\tilde{B}_t$

ii) $\int_0^t \sum \theta_i \xi \sigma_i \, d\tilde{B}_s$ will be Q-mart.

if X_t is complete.

- Note that for an attainable claim $F(\omega)$, which has represent: $\xi(T) F = z + \int_0^T \phi \, d\tilde{B}_t$. if we want to find portfolio θ to hedge it. θ will be: $\theta(t) = X_0(t) \phi(t) \Lambda(t)$.

So: next we want to find $\phi(t)$. given F.

<u>Thm.</u> For Itô diffusion $Y(t)$:

$$dY_t = b(Y_t) dt + \sigma(Y_t) d\tilde{B}_t. \quad Y(0) = \eta.$$

$h: \mathbb{R}^k \to \mathbb{R}$. s.t. $\left(\frac{\partial}{\partial \eta_i} \mathbb{E}_Q^\eta (h(Y(T-t)))\right)_1^k$ exists.

If $\mathbb{E}_Q^\eta \left(\int_0^T \phi^2 dt \right) < \infty$. where $\phi(t, \omega) =$

$\sum_{i=1}^k \frac{\partial}{\partial \eta_i} \mathbb{E}_Q^\eta (h(Y_{T-t})) \big|_{\eta = Y(t)} \sigma_i(Y(t))$. Then:

$$h(Y(T)) = \mathbb{E}_Q(h(Y_T)) + \int_0^T \phi(t) \, d\tilde{B}_t.$$

<u>Pf:</u> Set $g(t, \eta) =: \mathbb{E}_Q^\eta (h(Y(T-t)))$

$$Z(t) =: g(t, Y(t)).$$

By Kolmogorov backward equation:

$$\frac{\partial g}{\partial t} + \sum b_i \frac{\partial g}{\partial \eta_i} + \frac{1}{2} \sum (\sigma \sigma^T)_{ij} \frac{\partial^2 g}{\partial \eta_i \partial \eta_j} = 0.$$

By Itô: $dZ(t) = \sum \frac{\partial g}{\partial \eta_i}(t, Y_t) \sigma(Y_t) \, d\tilde{B}_t$.

$\Rightarrow$ Find $Z(0)$. $Z(T)$. $\Rightarrow h(Y(T)) = \cdots$

ii) $\bar{X}_t$ satisfies: $\begin{cases} d\bar{X}_0(t) = 0 \\ d\bar{X}_k(t) = \xi(t)\sigma_k(t)\, d\tilde{B}(t) \end{cases}$

if u satisfies $\sigma u = \mu - \rho X$.

So: $\bar{V}_t^{\theta}$ is also a local mart.

<u>Rmk</u>: Note $\tilde{\mathcal{F}}_t^{(m)}$ not necessarily equals to $\mathcal{F}_t^{(m)}$. $\Rightarrow$ i) isn't direct cor. of Itô's representation.

· Next. we assume $u(t,w)$ in the initial setting of Lemma exists. So: $(X_t)_{t \le T}$ has no arbitrage.

<u>Def</u>: i) A European T-claim is r.v. $F(w)$. st. $F \in \mathcal{F}_T^{(m)}$. $F \in L^2(Q)$. has lower bdd.

ii) Claim F is attainable if $\exists\, z \in \mathbb{R}'$ and θ admissible portfolio. st.

$F(w) = z + \int_0^T \theta(t)\, dX(t) =: V_\theta^z(t)$. a.s.

and $\bar{V}_\theta^z(t) =: z + \int_0^T \xi^{-1}(t)\theta(t)\, dX(t)$ is Q-mart. $0 \le t \le T$.

<u>Rmk</u>: If such $\theta(t)$ exists. We call it hedging portfolio of F. and z must be $\mathbb{E}_Q(F\xi(T))$.

iii) Market $(X_t)_{t \le T}$ is complete if

$\forall$ claim is attainable

Rmk: By def. X_t complete $\Rightarrow$ So

is $\bar{X}_t$. (normalization)

Thm. (Criteria)

$(X_t)_{t \le T}$ is complete $\Leftrightarrow$ σ has one

$\mathcal{F}_t^{(m)}$-adapted left inverse $\Lambda(\omega, t) \in \mathbb{R}^{m \times n}$

i.e. $\Lambda(t, \omega)\, \sigma(t, \omega) = I_m$. a.s.

Pf: $(\Leftarrow)$ $\xi(T) F(\omega) \in L^2(\Omega, \mathcal{F}_T^{(m)})$.

Then use Lemma. to represent it.

We can solve $\hat{\theta}(t, \omega) = (\theta_1 \cdots \theta_n)$

s.t. $\xi(t)\, \hat{\theta}(t)\, \sigma(t) = \phi(t)$.

$\Rightarrow$ Choose $\theta_0(t)$ s.t. θ is self-financing.

$(\Rightarrow)$ Choose $F(\omega) = \int_0^T \phi \, d\tilde{B}_t$. where

$\phi \in L^2(\Omega, [0, T])$. ϕ is $\mathcal{F}_t^{(m)}$-adapted.

We can also obtain $\exists\, \theta$. s.t.

$\hat{\theta} \sigma = \phi$. for $\forall$ such ϕ.

$\Rightarrow$ $r(\sigma) = m$. So Λ exists!

Cor. X_t is complete $\Rightarrow$ $\exists$ unique $u(t, \omega)$.

s.t. $\sigma(t, \omega)\, u(t, \omega) = \mu(t, \omega) - \rho(t, \omega)\, X_t(\omega)$.

Thm. Market X_t is complete $\Leftrightarrow$ There exists unique equi. mart. measure for $(\bar{X}_t)_{t \le T}$ normalized market.

(3) Option Pricing:

① European Options:

Def. A European option on claim F is a guarantee to be paid $F(\omega)$ at $t = T$.

Buyer:

Pay y to buy the option. To profit: $V_\phi^{-y}(T) + F \geq 0$. a.s.

Seller:

Receive z to sell the option. To profit: $V_\phi^{z}(T) - F \geq 0$. a.s.

So: $p(F) =: \sup\{y \mid \exists$ admissible portfolio ξ. s.t. $V_\xi^{-y}(T) + F \geq 0$. a.s.$\}$.

$q(F) =: \inf\{z \mid \exists$ admissible portfolio ψ. s.t. $V_\psi^{z}(T) - F \geq 0$. a.s.$\}$.

$p(F)$. $q(F)$ are resp. the max / min which buyer / seller can accept.

Def: i) If $p(F) = q(F)$. We call it common value, the price of T claim $F(\omega)$

ii) $F(\omega) = (X_i(T,\omega) - k)^+$. $k>0$ is called the European call.

iii) $F(\omega) = (k - X_i(T,\omega))^+$. $k>0$ is called the European put.

Rmk: European put / call permits the owner to sell / buy one unit i^{th} asset at price k (specified) at $t=T$.

Thm. For u and $\mathbb{Q}$ (def as before) exists and F is a T-claim. Then:

i) $\operatorname{ess\,inf} F(\omega) \le p(F) \le \mathbb{E}_{\mathbb{Q}}(\xi(T)F) \le q(F) \le \infty$

ii) X_t is complete $\Rightarrow$ $p(F) = \mathbb{E}_{\mathbb{Q}}(\xi(T)F) = q(F)$.

Pf: i) $\forall y \le \operatorname{ess\,inf} F$. $\exists\, \varphi$ admissible. $\varphi \equiv 0$.

satisfies: $V^{\varphi}_{-y}(T) \ge -F(\omega)$. a.s.

So: $p(F) \ge y \to \operatorname{ess\,inf} F$.

For other inequi. only need to notice.

$\int \sum \varphi_i \sigma\, d\widetilde{B}_t$ is lower bdd local $\mathbb{Q}$-mart

$\Rightarrow$ it's supermart.

Rmk: To apply on Finance.

First assume: $\begin{cases} dX_0(t) = \rho(X_t) X_0(t) dt \\ dX_k(t) = \mu_k(X_t) dt + \sigma_k(X_t) d\widetilde{B}_t \end{cases}$

Write $F(\omega)$ into $h_0(X(T))$

Set $h(t) = \xi(t) h_0(X(t))$

② American options:

The difference between ① and ② is:

American option permits buyer to choose any exercise time τ before time T.

Rmk: To be reasonable. τ is $\mathcal{F}_t^{(m)}$ - stopping time.

Def: American T-claim $F(t,\omega)$ is $\mathcal{F}_t^{(m)}$ - adapted. conti. a.s.-lower bdd. $t \le T$.

Buyer:

$V_{-y}^{\varphi}(\tau(\omega), \omega) + F(\tau(\omega), \omega) \overset{a.s.}{\geq} 0$ $\exists \tau$

Seller:

$V_z^{\psi}(t,\omega) - F(t,\omega) \overset{a.s.}{\geq} 0.\ \forall t.$

So: $p_A(F) =: \sup \{ y \mid \exists \tau(\omega).$ and φ. admissible portfolio. s.t. $V_{-y}^{\varphi}(\tau, \omega) + F(\tau, \omega) \geq 0$. a.s$\}$.

$q_A(F) =: \inf \{ z \mid \exists \psi$ admissible. $\forall t \le T$. s.t. $V_z^{\psi}(t,\omega) - F(t,\omega) \geq 0$. a.s.$\}$.

Thm. i) If $\sup_{\tau \le T} \overline{\mathbb{E}}_a(\xi(\tau) F(\tau)) < \infty$. Then:

$$P_A(F) \le \sup_{\tau \le T} \overline{\mathbb{E}}_a(\xi(\tau) F(\tau)) \le Q_A(F) \le \infty.$$

ii) If X_t is complete in addition to i).

Then: $P_A(F) \overset{1)}{=} \sup_{\tau \le T} \overline{\mathbb{E}}_a(\xi(\tau) F(\tau)) \overset{2)}{=} Q_A(F)$.

Pf: i) Process as before.

ii) 1) Set $F_k = F \wedge k$. $G_k = X_0(T) \xi(\tau) F_k(\tau)$

$\Rightarrow G_k$ is T-claim. using completeness

and Q-mart. as before. Let $k \to \infty$.

2) Set $S_t = \max_{t \le \tau \le T} \overline{\mathbb{E}}_a(\xi(\tau) F(\tau) \mid \mathcal{F}_t^{(m)})$

use Doob's decomp: $S_t = M_t - A_t$.

$M_t = Z + \int_0^t \varphi_s \, d\widetilde{B}_s = S_t + A_t \ge S_t$. $Z = S_0$

Choose $\hat{\theta}_t = X_0(t) \varphi(t) \Lambda(t)$ by complete.

$\Rightarrow Z + \int_0^t \hat{\theta} \, d\overline{X} = Z + \int_0^t \varphi_s \, d\widetilde{B}_s \ge S_t$.

So: $Z + \int_0^t \theta \, dX \ge F(t)$. $t \le T$.

Rmk: When the market is Itô diffusion

and $F(t) = h(X_t)$. It connects with

the optimal control problem and $P_A(F)$

is the optimal solution if X_t complete.

Cor. For American call $F(t) = (X_1(t) - K)^+$.

$P_A(F) = e^{-rT} \mathbb{E}_a((X_1(T) - K)^+)$. $X = (X_0, X_1)$ is B-S market.