Preliminary

(1) Conformal mapping:

Def: For $\phi : D \xrightarrow{\sim} \mathbb{D}$. conformal isomorphism.

i) $d_\phi(z, z') =: |\phi(z) - \phi(z')|$. metric on D.

ii) $\hat{D}$ is completion of D w.r.t d_ϕ.

We say $\delta D = \hat{D}/D$ by martin boundary.

Rmk: i) $\hat{D}$ is indept of choice of ϕ.

ii) ϕ can be extended to : $\hat{D} \to \overline{\mathbb{D}}$

uniquely. (a homeomorphism)

iii) If ∂D is Jordan. Then $\partial D = \delta D$.

Distortion estimate:

Def: $\mathcal{U} =: \{ f : \mathbb{D} \to D \mid 0 \in D.$ simply connected

$D \neq \mathbb{C}.$ $f(0) = 0.$ $f'(0) = 1 \}$.

Rmk: For $f \in \mathcal{U}$. $\Rightarrow f = z + \sum_{n \geq 2} a_n z^n$.

Prop. If $f \in \mathcal{U}$. Then $|a_2| \leq 2$.

Pf: Lemma. For $f \in \mathcal{S}$. $\exists h \in \mathcal{S}$. odd func.

s.t. $f(z^2) = h^2(z)$.

Pf: extend $\tilde{f}(z) = \begin{cases} f(z)/z. & z \neq 0 \\ f'(0). & z = 0. \end{cases}$

$\Rightarrow \tilde{f}(z) \neq 0$. conformal in $\mathbb{D}$.

$\therefore \tilde{f}(z) = e^{g(z)} = \tilde{g}^2(z)$.

Set $h(z) = z\tilde{g}(z^2)$. odd.

It's easy to check $h \in \mathcal{S}$

Note $h^2(z) = (z + a_3 z^3 + a_5 z^5 + \cdots)^2 =$

$f(z^2) = (z^2 + c_2 z^4 + c_3 z^6 + \cdots)$

$\Rightarrow a_3 = c_2/2$. Consider $\hat{g}(z) = 1/h(1/z)$.

$\hat{g}(z) = z(1 - \frac{c_2}{2} z^{-2} + \cdots) = z - \frac{c_2}{2} z^{-1} + \cdots$

The conclusion follows from the next prop.

Def: $\mathcal{K} = \{ K \subset \mathbb{C} \mid K$ is cpt. connected. $\mathbb{C}/K$ is connected.

$0 \in K$. $\lim_{z \to \infty} F(z)/z = 1$. where $F: \mathbb{C}/\overline{\mathbb{D}} \to \mathbb{C}/K$. conformal. isomorphism. $\}$.

Rmk: i) $F(z) = 1/f(1/z)$. where $f: \mathbb{D} \xrightarrow{\sim} I(\mathbb{C}/K)$

$I(z) = \frac{1}{z}$. f is conformal. isomorphism.

ii) $F(z) = z + b_0 + \sum_{n \geq 1} b_n / z^n$.

Prop. Using notation above. If $K \in \mathcal{U}$. Then

$$\text{Area}(K) = \pi(1 - \sum_{n\geq 1} n|b_n|^2) \geq 0.$$

Pf: For $K_r = F(r\bar{\mathbb{D}})$. $\gamma(\theta) = F(re^{i\theta})$

1) $\frac{1}{2i}\int_\gamma \bar{z}\,dz = \frac{1}{2i}\int_\gamma (x-iy)(dx+idy)$

$$\overset{\text{Green}}{=} \frac{1}{2i}\iint_{F(r\mathbb{D})} 2i\,dx\,dy$$

$$= \text{Area}(F(r\mathbb{D}))$$

2) $\frac{1}{2i}\int_\gamma \bar{z}\,dz = \frac{1}{2i}\int_0^{2\pi} \overline{F(re^{i\theta})}\, F'(re^{i\theta})\, ire^{i\theta}\,d\theta$

$$= \pi(r^2 - \sum n|b_n|^2 r^{-2n}).$$

Set $r \to 1^-$. (by Abel. Thm.)

Cor. $\forall f \in \mathcal{U}$. $f = \sum_{n\geq 1} a_n z^n$.

Then: $\text{Area}(f(\mathbb{D})) = \pi \sum_{n\geq 1} n|a_n|^2$.

Thm. (Koebe-$\frac{1}{4}$)

If $f \in \mathcal{U}$. $0 < r \leq 1$. Then $B(0, r/4) \subseteq f(r\mathbb{D})$.

Pf: WLOG. $r=1$. (Or set: $f(rz)/r$)

Suppose $f = z + \sum_{n\geq 2} a_n z^n : \mathbb{D} \to D$. Fix $z_0 \notin D$.

Set $\tilde{f}(z) = z_0 f(z) / (z_0 - f(z))$. $\in \mathcal{U}$.

expand $\tilde{f}$. $\Rightarrow$ So: $|a_2| \leq 2$. $|a_2 + \frac{1}{z_0}| \leq 2$

(2) Brownian and Harmonic:

① Thm. (Conformal Variance of BM)

$\phi: D \xrightarrow{\sim} D'$. conformal isomorphism. Fix $z \in D$. $\phi(z) = z'$. Let (B_t). (B_t') are 2 complex BM. starts at z. z'. respectively.

$T = \inf\{t \geq 0 \mid B_t \notin D\}$. $T' = \inf\{t \geq 0 \mid B_t' \notin D'\}$

Set $\varphi: s \mapsto \int_0^s |\phi'(B_r)|^2 dr$. $z(t) = \varphi^{-1}(t)$.

Then: $(\varphi(T), (\phi(B_{z(t)}))_{t<\varphi(T)}) \overset{d}{\sim}$

$(T', (B_t')_{t<T'})$

Pf: Only prove $(\phi(B_{z(t)})_{t<\varphi(T)}) \overset{d}{\sim} (B_t')_{t<T'}$.

Set $\phi(z) = u(z) + iV(z)$.

Consider $X_t = u(B_t)$. $Y_t = V(B_t)$.

are c.l.m. since $\Delta u = \Delta V = 0$.

With: $\langle X, Y\rangle_t = 0$. $[X]_t = [Y]_t = \int_0^t |\phi'(B_r)|^2$

prop. For BM $(\vec{B}_t)_{t \geq 0}$ starts at x. Set $z = \inf\{t \geq 0 \mid B_t \notin B(x,r)\}$. Then we have:

$B_z \sim \sigma_r(x,\cdot)$. uniform dist. on $\partial B(x,r)$

Pf: Lemma. (characterization of uniform dist.)

If M is p.m. on $\partial B(x,r)$. st.

invariant under orthogonal linear transf. on $\mathbb{R}^n$. Then $\mu = \sigma_r(x)$.

$\Rightarrow$ Note (B_t) is isotropy as well.

<u>prop.</u> D is proper simply connected. Fix $z \in D$. (B_t) is complex BM starts at z. For $T(D) = \inf\{t \geq 0 \mid B_t \notin D\}$. $\Rightarrow \mathbb{P}_z(T(D) < \infty) = 1$.

<u>Pf:</u> $\phi: D \xrightarrow{\sim} \mathbb{D}$. conformal isomorphism.

$(\phi(B_t))_{t < T(D)}$ is time-change BM.

If $T(D) = \infty$. Note as $t \uparrow T(D)$.

$|\phi(B_t)| \geq \frac{1}{2}$ $(\phi(\partial D) = \partial \mathbb{D})$

But it's recurrent $\Rightarrow$ visits $\{|\phi| < \frac{1}{2}\}$ at an unbdd set of times!

<u>Thm.</u> (Kakutani's Formula)

u is harmonic on bdd domain $\bar{D}$. Fix $z \in D$ (B_t) is complex BM. start at z. $T(D) = \inf\{t \geq 0 \mid B_t \notin D\}$. Then $u(z) = \mathbb{E}_z(u(B_{T(D)}))$

<u>Pf:</u> By Itô's Formula on $u(B_t)$.

<u>Lemma.</u> (Estimate of Harmonic Func.)

u is harmonic in $D \Rightarrow \left|\frac{\partial u}{\partial x}(z)\right| \leq \frac{4\|u\|_\infty}{\pi\, d(z, \partial D)}$. $\forall z \in D$.

② Harmonic Measure:

Thm. (First Exit distribution)

i) For $\tilde{z} \in \mathbb{D}$. the first exit dist of complex BM starts from $\tilde{z}$ at $e^{i\theta}$ is

$$h_{\mathbb{D}}(\tilde{z}, \theta) = \frac{1}{2\pi} \frac{1-|\tilde{z}|^2}{|e^{i\theta}-\tilde{z}|^2}$$

ii) For $\tilde{z} = x+iy \in \mathbb{H}$. the first exit dist. of complex BM. starts from $\tilde{z}$ at u is $h_{\mathbb{H}}(\tilde{z}, u) = \frac{1}{\pi} \cdot \frac{y}{(x-u)^2+y^2}$

Pf: i) $\phi: \mathbb{D} \xrightarrow{\sim} \mathbb{D}$ $\qquad \phi(0) = \tilde{z}$.

$$w \mapsto \frac{w+\tilde{z}}{1+w\bar{\tilde{z}}}$$

Note that $\mathbb{P}^0(B_{T(\mathbb{D})} \in dz)$

$$= \frac{1}{2\pi} dz = \mathbb{P}^{\phi(0)}(\phi(B_{T(\mathbb{D})}) \in d\phi(z))$$

$$\overset{(*)}{=} \mathbb{P}^{\tilde{z}}(\tilde{B}_{T(\mathbb{D})} \in d\phi(z)) \quad . \quad z = \phi^{-1}(e^{i\theta})$$

by conformal invariance of BM. on $(*)$.

$$\Rightarrow h_{\mathbb{D}} \sim \frac{1}{2\pi} |(\phi^{-1}(z))'| = 1/2\pi |\phi'|_{\phi^{-1}(e^{i\theta})}$$

ii) Set $\phi: \mathbb{D} \to \mathbb{H}$. st. $\phi(0) = \tilde{z}$.

Def: For D domain with martin boundary δD

As $t \uparrow T(D)$. $B_t \to \hat{B}_T \in \delta D$ in $\hat{D}$. We say $h_D(z, \cdot)$ is harmonic measure for D

starts at z. if it's dist. of $\hat{B}_\tau$ on δD. where B_t is complex Bm starts at z.

Rmk: By Kakutani's Formula:

$$u(z) = \mathbb{E}_z(u(\hat{B}_\tau)) = \int_{\delta D} u(s)\, h_D(z, ds)$$

prop. $h_D(z, dw) \sim \frac{1}{2\pi} \cdot \frac{1}{|\phi'||_{\phi^{-1}(w)}} dw$

Pf: generalize the pf above.

③ Green Function:

prop. For $(X_s)_{0\le s\le t}$ Brownian bridge from X to η in time t. Then $\pi_D(t, X, \eta) =: \mathbb{P}(X_s \in D.\ \forall s \in [0,t])$ is symmetric and joint conti. on (X, η).

Pf: Only prove sym:

For $(W_s)_{0\le s\le 1}$. Brownian bridge in $\mathbb{R}^2$ from 0 to 0. in time 1.

$$\Rightarrow X_s = (1-\frac{s}{t})X + \frac{s}{t}\cdot\eta + \sqrt{t}\, W_{s/t}$$

Def: i) Dirichlet heat kernel P_D on $(0,\infty) \times D^2$ is

$$P_D(t, X, \eta) = p(t, X, \eta) \cdot \pi_D(t, X, \eta). \text{ where}$$

$$p(t, X, \eta) = e^{-|X-\eta|^2/2t} / 2\pi t.$$

ii) Green func. $G_D(x,\eta) = \int_0^\infty P_D(t,x,\eta)\, dt$.

<u>Rmk</u>: i) Note $\pi_D(t,x,x) \xrightarrow{t\to 0} 1 \Rightarrow G_D(x,x) = \infty$.

ii) $\int_D G_D(x,\eta) f(\eta)\, dm(\eta) = \int_D \int_0^\infty p\, \pi_D f(\eta)\, dm(\eta)$

$= \mathbb{E}_x (\int_0^\infty f(B_t) I_{\{t<T(D)\}}\, dt)$

$= \mathbb{E}_x (\int_0^{T(D)} f(B_t)\, dt)$. m is Lebesgue on $\mathbb{C}$.

iii) $D \subseteq \mathbb{C}$ is Greenian if $G_D(x,\eta) < \infty$ for $\forall x, \eta \in D$.

<u>prop.</u> i) $\forall$ bdd domain is Greenian.

ii) In Greenian domain D. G_D is finite and conti. on $D \times D / \Delta$. $\Delta = \{(x,x) \mid x \in D\}$.

<u>prop.</u> (Conformal Invariance)

If $\phi : D \xrightarrow{\sim} \phi(D)$. conformal isomorphism.

Then $G_{\phi(D)}(\phi(x), \phi(\eta)) = G_D(x,\eta)$. $x, \eta \in D$.

<u>Pf</u>: Check: $\int_D G_{\phi(D)}(\phi(x), \phi(\eta))\, g(\eta)\, dm(\eta) = \int_D G_D(x,\eta)\, g(\eta)\, dm(\eta)$. $\forall g = (f\circ\phi)|\phi'|^2$. $\forall f \in C_c^\infty$.

follow from conformal invariance of BM.

<u>Rmk</u>: Note $P_{\mathbb{H}}(t,x,\eta) = p(t,x,\eta) - p(t,\bar{x},\eta)$ by reflec. prin. $\Rightarrow G_{\mathbb{H}}(x,\eta) = \frac{1}{\pi} \log \left| \frac{\eta - \bar{x}}{\eta - x} \right|$. $G_{\mathbb{D}}(0,\eta) = -\frac{\log|\eta|}{\pi}$

Def: (Another definition for green func.)

i) $H_\eta(x) =: \frac{1}{2\pi} \cdot \log|x-\eta|^{-1}$. on $\mathbb{R}^2/\{\eta\}$.

ii) $h_{\eta,D}(x) =: \mathbb{E}_x(H_\eta(B_{T_D}))$

iii) Set $\widetilde{G}_D(x,\eta) =: 2(H_\eta(x) - h_{\eta,D}(x))$

Lemma. $\widetilde{G}_D = G_D$. a.s.

Lemma. (Characterization)

$\forall \eta \in D$. $x \mapsto \widetilde{G}_D(x,\eta)$ is unique conti. func in $\bar{D}/\{\eta\}$. s.t. i) $\widetilde{G}_D(\cdot,\eta) = 0$ on ∂D

ii) $\Delta\widetilde{G}_D(\cdot,\eta) = 0$ in $D/\{\eta\}$.

iii) $x \mapsto \widetilde{G}_D(x,\eta) - H_\eta(x)$ is bdd on the nbd of η.

prop. $\widetilde{G}_D(f)(x) =: \int_D f(\eta)\widetilde{G}_D(x,\eta)\,d\eta$. defined for $f \in C_c(D)$.

$\Rightarrow$ $\widetilde{G}_D(f) \in C(\bar{D}) \cap C^\infty(D)$. Vanish on ∂D. $\frac{-\Delta}{2}\widetilde{G}_D(f) = f$

Rmk: $\widetilde{G}_D = G_D$ is inverse of $\frac{-1}{2}\Delta$.