Gaussian Measure Theory

Setting: B is separable Banach space.

All measures we consider are Borel.

(1) Definitions:

Def: Gaussian p.m. on B is Borel measure s.t. $l^*\mu$ (push-forward) is Gaussian p.m. on $\mathbb{R}'$ for $\forall l \in B^*$.

Rmk: A natural def for mean is:

$$m(l) =: \int_B l(x)\,\mu(dx). \quad \forall l \in B^*.$$

Since we don't know $x \to \|x\|$ is integrable or not. So $m =: \int_B x\,\mu(dx)$ seems not to be well-def.

prop. For μ, ν Gaussian p.m's on B. If $l^*\mu = l^*\nu$. $\forall l \in B^*$. Then $\mu = \nu$.

Pf: Set $cyl(B) =: \{ A \mid \exists \hat{A} \in B_{\mathbb{R}'}.\ l \in B^*.\ A = l^{-1}(\hat{A}) \}$.

Note $\mu(A) = \nu(A)$. $\forall A \in cyl(B)$.

Set $\Sigma(B) = \sigma(cyl(B))$. Next:

prove: $\Sigma(B)$ = σ-algebra of B.

i.e. $\forall$ open set $\in \Sigma(B)$.

Note U. open $\Rightarrow U = \bigcup \bar{B}(x_n, r_n)$. by separa.

Check: $\bar{B}(0,1) \in \Sigma(B)$.

It follows from. $\bar{B}(0,1) = \bigcap_{n \geq 0} \{x \in B \mid |l_n(x)| \leq 1\}$.

where $l_n \in B^*$. $\|l_n\| = 1$. $l_n(x_n) = 1$.

Def: Given centered Gaussian p.m μ. on B.

$C_\mu : B^* \times B^* \to \mathbb{R}$ is defined by:

$C_\mu(l, l') = \int_B l(x) l'(x) \mu(dx)$. covar. opera.

Rmk: i) Another perspective:

$\tilde{C}_\mu : B^* \to B^{**}$. $\hat{C}_\mu(l)(l') =:$

$C_\mu(l, l')$

ii) $\hat{\mu}(l) =: \int_B e^{i l(x)} \mu(dx) = e^{-\frac{1}{2} C_\mu(l,l)}$

Fourier transf. of μ.

prop. μ, ν. two p.m.'s on B. If $\hat{\mu}(l) = \hat{\nu}(l)$.

for $\forall l \in B^*$. Then $\mu = \nu$.

Pf: By def. consider $B = \mathbb{R}^n$.

$\int_B \varphi(x) \mu(dx) = \int_{\mathbb{R}^n} \hat{\varphi}(y) \hat{\mu}(y) dy$. for

$\forall \varphi \in C_B(\mathbb{R}^n)$.

Cor. μ is Gaussian p.m. on B. For $\varphi \in \mathbb{R}^1$. Def $R_\varphi : B^2 \longrightarrow B^2$.

$$(x,y) \longmapsto \begin{pmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow R_\varphi^*(\mu \otimes \mu) = \mu \otimes \mu.$$

Pf: Check: $\widehat{\mu \otimes \mu} \circ R_\varphi = \widehat{\mu \otimes \mu}$.

Thm. (Fernique)

$\forall \mu$ on B. finite measure. satisfies the conclusion of cor. above for $\varphi = \pi/4$. Then:

$\exists \alpha > 0$. s.t. $\int_B e^{\alpha \|x\|^2} \mu(dx) < \infty$.

Cor. $\|C_\mu\| < \infty$ for Gaussian p.m. μ.

Pf: Note $\int_B \|x\|^2 \mu(dx) < \infty$.

$$\Rightarrow |C_\mu(\ell, \ell')| \le \|\ell\| \|\ell'\| \int_B \|x\|^2 d\mu.$$

Cor. $\tilde{C}_\mu$ is BLO from B^* to B.

Pf: Check: $R(\tilde{C}_\mu) \subset B$.

Since $\tilde{C}_\mu(\ell) = \int_B x\, \ell(x)\, d\mu$. well-def.

Rmk: THM holds for $\forall \alpha < 1/2\|C_\mu\|$.

Next, we consider properties of Gaussian p.m. μ.

prop. There exists $\alpha, k > 0$. s.t. $\forall f: \mathbb{R}^+ \to \mathbb{R}^+$ measurable. s.t. $f(x) \le C_f e^{\alpha x^2}$. $\forall x \ge 0$. Then:

$$\int_B f(\|x\| / \int_B \|x\| d\mu) \, \mu(dx) \le k C_f.$$

Cor. Set $f = e^{\alpha x^2}$. Then:

$$\int_B \|x\|^{2n} d\mu \le n! \cdot k \alpha^{-n} \cdot (\int_B \|x\| d\mu)^{2n}$$

Rmk: Any k^{th}-moment can be dominated by power of first moment.

prop. (Characterization in Hilbert space)

In $B = H$. Hilbert space. $\hat{C}_\mu$ is a trace class operator. s.t. $tr(\hat{C}_\mu) = \int_B \|x\|^2 d\mu$.

Conversely. $K: H \to H$. is sym. positive trace class operator. Then $\exists$ Gaussian p.m. μ on H. s.t. $K = \hat{C}_\mu$.

Pf: 1) Find (e_n) o.n.b. of H. easy check.

2) K is cpt. normal. $\Rightarrow \exists (e_n)$. s.t.

$K e_n = \lambda_n e_n$. $\Sigma \lambda_n < \infty$. $\lambda_n > 0$.

Set $P_n = \sum_1^n \sqrt{\lambda_k} \xi_k e_k$. $(\xi_k) \overset{i.i.d}{\sim} N(0,1)$

P_n converges in L^2. $\Rightarrow \exists P = \lim_k P_{n_k}$. a.s.

Take μ is law of $P^* = P$.

Prop. $\hat{C}_m$ is cpt operator in general B.

Pf: By contradiction

Thm. (Kolmogorov continuity criterion)

For $d \geq 1$. $C: [0,1]^d \times [0,1]^d \to \mathbb{R}'$. sym. st.

$\forall (x_j)_1^n \subset [0,1]^d$. $(C(x_i, x_j))_{n \times n}$ is positive-definite.

If $\exists \alpha, k > 0$. st. $C(x,x) + C(y,y) - 2C(x,y) \leq$

$k|x-y|^\alpha$. $\forall x, y \in [0,1]^d$. Then:

$\exists$ unique centered Gaussian p.m. μ on

$C([0,1]^d, \mathbb{R}')$. st. $C(x,y) = \int_{C([0,1]^d, \mathbb{R}')} f(x) f(y) \mu(df)$.

Besides. $\mu(C^\beta([0,1]^d, \mathbb{R}')) = 1$. $\forall \beta < \alpha$.

Thm. For $d \geq 1$. $C: [0,1]^d \times [0,1]^d \to \mathcal{L}(H, H)$. where

H is Hilbert. st. positive trace class sym.

and $tr(C(x,x) + C(y,y) - 2C(x,y)) \leq k|x-y|^\alpha$.

$\Rightarrow \exists \mu$ st. $\int_{C^\beta([0,1]^d, \mathbb{R}')} \langle h, f(x)\rangle \langle f(y), k\rangle d\mu(f)$

$= \langle h, C(x,y) k\rangle$. $\forall \beta < \alpha$. $h, k \in H$. $x, y \in [0,1]^d$

Thm. For $(X(x))_{x \in [0,1]^d}$. B-valued. Gaussian r.v.'s.

st. $\exists C > 0$. $\alpha \in (0,1]$. st. $\mathbb{E}\|X_x - X_y\| \leq C|x-y|^\alpha$.

$\Rightarrow \exists$ unique Gaussian measure μ on $C([0,1]^d, \mathbb{R}')$.

st. $Y \sim \mu$. Then. $X \overset{law}{=} Y$. and $\mu(C^\beta([0,1]^d, \mathbb{R}')) = 1$. $\forall \beta < \alpha$.

(2) Cameron - Martin space:

① Def: CM space H_μ of μ is completion of $\mathring{H}_\mu =: \{h \in B \mid \exists h^* \in B^*, \text{ s.t. } C_\mu(h^*, l) = l(h)$ for $\forall l \in B^*\}$. under norm: $\|h\|_\mu^2 =: C_\mu(h^*, h^*)$

Rmk: $h \mapsto h^*$. may not be one-to-one:

e.g. Set $\mu = \delta$. $h = 0$. Then $\forall l \in B^*$ ✓.

But $\|h\|_\mu^2$ is exactly well-def:

If $h \in \mathring{H}_\mu \to h_1^*, h_2^*$. Set $k = h_1^* + h_2^*$

$\Rightarrow C_\mu(h_1^*, h_1^*) - C_\mu(h_2^*, h_2^*) =$

$C_\mu(h_1^*, k) - C_\mu(h_2^*, k) = k(h) - k(h) = 0$.

Besides, it's injective. by def.

prop. In $B = H$. Hilbert. μ is Gaussian p.m. on B. with cov K. st. $\exists (e_n)$. o.n.b. $Ke_n = \lambda_n e_n$. $\Sigma \lambda_n < \infty$. $\lambda_n > 0$. Then:

i) $R(K) = \mathring{H}_\mu$. $h \mapsto h^*$ is given by $h^* = K^{-1}h$.

ii) $\langle h, g \rangle_\mu = \langle K^{-\frac{1}{2}} h, K^{-\frac{1}{2}} g \rangle$. $H_\mu = \{h \in B \mid \Sigma \lambda_n^{-1} \cdot \langle h, e_n \rangle^2 < \infty\}$.

Pf: i) By Riesz. Thm. ii) follows from i)

Rmk: We can see $\mathring{H}_\mu$ as range of $\hat{C}_\mu$.

Prop. μ, ν are two Gaussian measures on B.

If $H_\mu = H_\nu$. $\|h\|_\nu = \|h\|_\mu$. $\forall h \in H_\mu$.

Then: $\mu = \nu$.

Rmk: $H_\mu \subset B$. So it's stronger than charac. by using chf.

prop. $\langle h, h \rangle_\mu \geq \|h\|^2 / \|C_\mu\|$. So: $H_\mu \subset B$.

Pf: $\|h\|^2 = \sup_{\|\ell\|=1} \ell(h)^2 = \sup_{\|\ell\|=1} C_\mu(h^*, \ell)^2 \leq \cdots$

prop. There's a canonical isomorphism $\iota: h \longmapsto h^*$. between H_μ and closure R_μ of B^* in $L^2(B, \mu)$

Cor. H_μ is separable. Hilbert.

Pf: $\|h\|_\mu^2 = C_\mu(h^*, h^*) = \int_B h^*(x)^2 \, d\mu(x)$

Besides. $\forall h^* \in B^*$. set:

$h = \int_B x \, h^*(x) \, d\mu(x)$. $\Rightarrow$ $h \in \mathring{H}_\mu$. $h^* = \iota(h)$.

Rmk: i) Note $h_1 = h_2$ only require a.e. hold. The counter-example also holds in this sense

ii) We call R_μ reproducing kernel Hilbert space of μ.

Cor. $\|h\|_M = \sup\{l(h) \mid \langle m(l, l) \le 1\}$.

$H_M = \{h \in B \mid \|h\|_M < \infty\}$.

Pf: equip inner product $\langle_M \cdot, \cdot\rangle$ in H_M. By prop. above.

② **prop.** $\forall l \in R_M$. There exists a measurable linear subspace V_l of B. and $\hat{l}$ is linear on V_l. s.t. $m(V_l) = 1$. $\hat{l} = l$ a.s.

Pf: We have $(l_n) \subset B^*$. $l_n \to l$ a.s.

Set $V_l = \{\lim_{n\to\infty} l_n \text{ exists}\}$. and

$\hat{l} = \lim_{n\to\infty} l_n$ on V_l.

Rmk: Converse is true: if $l: B \to \mathbb{R}'$ is measurable and linear on a measurable linear subspace V of full measure. $\Rightarrow l \in R_M$.

prop. $h^* = l(h) \in R_M$ is centered Gaussian with Var $\|h\|_M^2$. Besides. $\mathrm{cov}(h^*, k^*) = \langle h, k\rangle_M$.

Pf: Find $h_n \in B^* \to h^*$. Set:

$\tilde{h}_n = \frac{\|h^*\|}{\|h_n\|} h_n \sim N(0, \|h^*\|^2) \to h^*$. a.s.

So: $h^* \sim N(0, \|h\|_M^2)$.

(3) Prop. μ is center Gaussian on B. If $\dim H_\mu = \infty$. Set $D_c : \begin{matrix} B \to B \\ x \mapsto cx \end{matrix}$. $c \in \mathbb{R}'$. Then:

$\mu \perp D_c^* \mu$. $\forall c \neq \pm 1$.

Pf: Set (l_n) is o.n.b. of R_μ.

$$X_N(x) =: \frac{1}{N} \sum_1^N |l_n(x)|^2.$$

By SLLN: $X_N \to 1$. μ-a.s.

$X_N \to c^2$. $D_c^* \mu$-a.s.

Thm. (Cameron-Martin).

For $h \in B$. $T_h : \begin{matrix} B \to B \\ x \mapsto x+h \end{matrix}$. Then:

$T_h^* \mu \ll \mu \iff h \in H_\mu$.

Pf: $(\Leftarrow)$. Set: $h^* = l(h) \in R_\mu$. $D_h(x) = e^{h^*(x) - \frac{1}{2}\|h\|_\mu^2}$.

$d\mu_h =: D_h \cdot d\mu \ll d\mu$.

check: $\mu_h = T_h^* \mu$ by Fourier transf.

$(\Rightarrow)$ i) Check: $\| N(0,1) - N(h,1) \|_{TV}$

$\geqslant 2 - 2e^{-h^2/8}$.

ii) If $h \notin H_\mu$. Then $\exists\, l$. s.t.

$l(h) \geqslant n$. $C_\mu(l,l) = 1$.

So: $\|\mu - T_h^*\mu\|_{TV} \geqslant \|l^*\mu - l^* T_h^* \mu\|_{TV}$

$= \| N(0,1) - N(-l(h), 1)\|_{TV}$

$\geqslant 2 - 2e^{-n^2/8} \xrightarrow[n\to\infty]{} 2$

$\Rightarrow \mu \perp T_h^* \mu$.

Prop. (Characterization of H_μ)

H_μ is intersection of all measurable linear subspace of full measure.

Besides, if $\dim H_\mu = \infty$, then: $\mu(H_\mu) = 0$.

Pf: i) $\forall V$. linear subspace. $\mu(V) = 1$.

$\forall h \in H_\mu$. By CM Thm:

$\mu(V-h) = \mu(V) = 1 \Rightarrow V \cap (V-h) \neq \emptyset$.

So: $h \in V$. $\Rightarrow H_\mu \subset V$. $H_\mu \subset \bigcap_{\{\dots\}} V$

$\forall x \notin H_\mu$. $\|x\|_\mu = \infty$.

i.e. $\exists (l_n)$. $C_\mu(l_n, l_n) = 1$. $l_n(x) \geq n$.

Set $|\eta| = \sum^\infty (l_n(\eta))^2/n^2$.

$\tilde{V} = \{|\eta| < \infty\}$. $\Rightarrow x \notin \tilde{V}$.

But $\int_B |\eta|^2 d\mu = \sum 1/n^2 < \infty \Rightarrow \mu(\tilde{V}) = 1$.

2) Find $(l_n(x)) \overset{i.i.d}{\sim} N(0,1)$. o.n.b. of H_μ.

By SLLN. $\|x\|_\mu^2 \geq \sum_1^\infty l_n^2(x) = \infty$

for μ-a.s. $x \in B$.

(3) Image of Gaussian measure.

Thm. μ is centered Gaussian on B. H is separable Hilbert. $A \in \mathcal{L}_{HS}(H_\mu, H)$. Then. $\exists \hat{A}: B \to H$. measurable. s.t. $\nu = \hat{A}^*\mu$ is Gaussian on H. and

$C_\nu(h,k) = \langle A^* h, A^* k\rangle_\mu$.

Besides. $\exists$ measurable linear subspace $V \subset B$.

s.t. $\mu(V) = 1$. $\hat{A}|_V$ is linear. $H_\mu \subset V$ and

$$\hat{A}|_{H_\mu} = A$$

Pf: Set (e_n) o.n.b. of H_μ and correspd

to (e_n^*) o.n.b. of $R_\mu \subset L^2(B, \mu)$.

Def: $S_N(x) = \sum_1^{N} e_n^*(x) A e_n$

$$V =: \{ x \in \bigcap_{e_n} V_{e_n} \mid \lim_{N \to \infty} S_N(x) \text{ exists} \}.$$

where e_n^* is linear on V_{e_n}. $\mu(V_{e_n}) = 1$.

Note: S_N is H-valued mart. ((e_n) i.i.d)

and $\mathbb{E}_\mu(\|S_N\|^2) \le tr A^* A < \infty$

$\Rightarrow$ $\mu(V) = 1$. Set $\hat{A} = \begin{cases} 0 & B \setminus V \\ \lim_{N \to \infty} S_N & V \end{cases}$

Thm (Converse)

μ is Gaussian on B. $A \in \mathcal{L}_{HS}(H_\mu, \tilde{B})$ where $\tilde{B}$ is separable Banach. Then the linear measurable extension $\hat{A}$ of A on B is unique up to set of measure 0.

Rmk: It's intuitive since H_μ - the null measure space can determines a measurable map on full measure set

Pf: It follows from the Theorem below.

Thm. (Borell - Sudakov - Cirelson)

If $A \subset B$. measurable. st. $\mu(A) = \Phi(\tau)$.

Then. $\forall \varepsilon > 0$. $\mu(A + B_{H_\mu}(0,\varepsilon)) \geq \Phi(\tau + \varepsilon)$

where $\Phi \sim N(0,1)$.

Cor. $\mu(A) > 0 \Rightarrow \mu(A + H_\mu) = 1$.

Pf: Let $\varepsilon \to \infty$.

Cor. (Zero-one law).

$V \subset B$. measurable linear subspace

$\Rightarrow \mu(V) \in \{0,1\}$.

Pf: 1) $H_\mu \not\subset V$. Note that:

$$\sum_{x \in H_\mu} \mu(A + x) = \mu(A + H_\mu).$$

$$\Rightarrow \mu(A) = 0.$$

2) $H_\mu \subset V$. Then:

$$\mu(V + B_{H_\mu}(0,\varepsilon)) = \mu(V).$$

Prop. $A : H_\mu \to B_2$. BLO to separable Banach.

st. $\exists\, \nu$. Gaussian measure in B_2. with

$C_\nu(h,k) = \langle A^* h, A^* k \rangle_\mu$. $h,k \in H_\mu$. Then:

$\exists\, \hat{A} : B \to B_2$. measurable. st. $\nu = \hat{A}^* \mu$.

and full-measure linear subspace U. st.

$\hat{A}|_U$ is linear. $H_\mu \subset U$. $\hat{A}|_{H_\mu} = A$.

(4) Cylindrical Wiener process:

① Note that $C(\mathbb{R}^1, \mathbb{R}^1)$ isn't Banach. We consider to construct Wiener measure on separable Banach space $C_\ell(\mathbb{R}_+, \mathbb{R}^1) =: \{f \in C(\mathbb{R}_+) \mid \lim_{t\to\infty} \frac{f(t)}{\ell(t)} \text{ exists}\}$.

with norm $\|f\|_\ell = \sup_t |f|/\ell$. $\ell : \mathbb{R} \to \mathbb{R}^{\geq 1}$.

Denote: $CW =: C_{\ell = t^2+1}$.

Prop. There exists Gaussian measure μ on CW with cov $C(t,s) = t \wedge s$.

Pf: Set μ_0 is measure on $C([0,\pi], \mathbb{R}^1)$ with cov. $C_0(x,y) = \dfrac{\tan x \wedge \tan y}{(1+\tan^2 x)(1+\tan^2 y)}$

$Tf =: (1+t^2) f(\arctan t)$

Note $f \in CW \Leftrightarrow Tf \in C[0,\pi]$.

Check $T^* \mu_0$ satisfies Kolmogorov conti. Thm. and is what we need.

Fix separable Hilbert space H and H'. St. we have $H \underset{\text{dense}}{\subset} H'$. $\iota : H \hookrightarrow H'$ is HS-operator.

Rmk: For example, set $\|x\|_{H'}^2 =: \sum \frac{1}{n^2} \langle x, e_n \rangle^2$

So: $\iota\iota^t e_n = \frac{1}{n^2} e_n$.

Def: Cylindrical Wiener process on $\mathcal{H}$ is any $\mathcal{H}'$-valued Gaussian process W s.t. $\mathbb{E}(\langle h, W_s\rangle_{\mathcal{H}'} \langle W_t, k\rangle_{\mathcal{H}'}) = (s \wedge t) \langle \iota^* h, \iota^* k\rangle$.

Rmk: i) We can realize it as canonical process for some Gaussian measure on $C_W(\mathbb{R}^+, \mathcal{H}')$ by Kolmogorov conti.

Set $\tilde{C}(x,y) = C(x,y) \cdot \iota\iota^*$.

ii) It's not a $\mathcal{H}$-valued process. If we assume $(W_t) \subset \mathcal{H}$. Then:

$\mathbb{E}\langle h, W_t\rangle \langle k, W_s\rangle = t \wedge s \langle h, k\rangle_{\mathcal{H}}$

$\Rightarrow$ W_t will be thought as $\mathcal{H}$-valued r.v. with cov. tI ∉ trace class.

iii) Generally. (e_k) is o.n.b of $\mathcal{H}$. (B_t^k) is scalar SBM. Then we set: $W_t = \sum_{k \geq 1} e_k B_t^k$.

It's only convergent in $\mathcal{H}'$. with $\langle\langle W\rangle\rangle_t = tI$.

Prop: Gaussian measure μ on $\mathcal{H}'$ with cov. $\iota\iota^*$ has $\mathcal{H}$ as its CM-space.

Besides. $\|h\|_\mu^2 = \|h\|^2$. $\forall h \in \mathcal{H}$.

Rmk: For $A: H \to K$, $A \in \mathcal{L}_{HS}(H, K)$.

Since the extension of A on H'

only depend on $H'_m = H$.

So: $\hat{A} W(t)$ is well-def indept

of choice of H'.

Pf: Note that $\mathring{H}_m = R(LL^*)$ and $h^* = (LL^*)^{-1} h$.

$\forall h, k \in \mathring{H}_m$, $\exists \hat{h}, \hat{k} \in H$ s.t. $h = L\hat{h}$, $k = L\hat{k}$

$\langle h, k \rangle_m = \langle \hat{h}, L^*(LL^*)^{-1} L \hat{k} \rangle = \langle \hat{h}, \hat{k} \rangle$

② Consider W_t is cylindrical Wiener process

on $H \subset H'$. realized on $\Omega = C_0(\mathbb{R}^+, H')$.

with $\mathcal{F}_s =: \sigma(W_r, r \leq s)$.

Def: i) For $\{(s_n, t_n]\}$ disjoint $\subset \mathbb{R}^+$, $\phi_n \in \mathcal{F}_{s_n}$

$: \Omega \to \mathcal{L}_{HS}(H, K)$. where K is fixed

Hilbert. The elementary process ϕ is:

$$\phi(t, \omega) =: \sum_1^N \phi_n(\omega) I_{(s_n, t_n]}(t).$$

ii) K-valued stochastic integral of ϕ is:

$$\int_0^\infty \phi(t)\, dW_t =: \sum_1^N \phi_n(W)(W_{t_n} - W_{s_n})$$

Prop. $\mathbb{E}(\|\int_0^\infty \phi(t)\, dW_t\|_K^2) = \mathbb{E}(\int_0^\infty \mathrm{tr}\, \phi(t)\phi(t)^*\, dt)$.

for $\forall\ \phi$ elementary process above.

Pf: Set $W_{t_n} - W_{s_n} = \sum_k \langle W_{t_n} - W_{s_n}, e_k \rangle e_k$

and apply the def of W_t.

cor. The stochastic integral is isometry

from $L^2_{pred}(\mathbb{R}^+ \times \Omega, \mathcal{L}_2(H, K))$ to $L^2(\Omega, K)$

Lemma The set of elementary processes is

dense in $L^2_{pred}(\mathbb{R}_+ \times \Omega, \mathcal{L}_2(H, K))$.

Pf: Approxi. by simple func. use MCT.

cor. We can define $\int \varphi(t) dW_t$ unique

approxi. by elementary processes.

for $\forall \varphi \in L^2_{pred}(\mathbb{R}_+ \times \Omega, \mathcal{L}_2(H, K))$.