Stochastic Filtering

Consider SDE:

$dX_t = B(X_t)dt + C(X_t)dW_t \in \mathbb{R}^d$. (system)

with B, C global Lip. conti. W_t is BM.

And the info. on X_t is provided by

$dY_t = H(X_t)dt + \Gamma dV_t \in \mathbb{R}^p$. $Y_0 = 0$ (observation)

with H global Lip. V_t is p-dim BM indept of W. $R = \Gamma\Gamma^T > 0$.

Denote $\mathcal{Y}_{0,t} := \sigma(Y_s, s \le t)$. We want to compute / approxi. $\eta_t(A) := \mathbb{P}(X_t \in A \mid \mathcal{Y}_{0,t})$. $A \in \mathcal{B}_{\mathbb{R}^d}$.

(1) Kallianpur - Striebel formula:

Note that $\int_0^t \langle R^{-1}H(X_s), dY_s \rangle$ is well-def by $\mathbb{E}(\int_0^T \|X_s\|^2 ds) < \infty$ (Sol. for SDE)

Denote $$Z_t(X, Y) := \exp\Big(\int_0^t \langle R^{-1}H(X_s), dY_s\rangle - \frac{1}{2}\int_0^t \langle R^{-1}H(X_s), H(X_s)\rangle ds\Big)$$

Thm. $(\bar{X}_t)$ is copy of X_t. s.t. indept of (X, V). Then:

$\eta_t(A) = \mathbb{E}(I_A(X_t) \mid \mathcal{Y}_{0..t})$

$= \mathbb{E}_{\bar{X}}(I_A(\bar{X}_t) Z_t(\bar{X}, Y)) \Big/ \mathbb{E}_{\bar{X}}(Z_t(\bar{X}, Y))$

Rmk: We see $Y_t \mapsto \eta_t(A)$ isn't conti. w.r.t. $\|\cdot\|_\infty$ if $d > 1$. $\Rightarrow$ Z_t's not robust. $\Rightarrow$ Rough path comes in.

Pf: $\mathbb{P}_{(X,Y)} \ll \mathbb{P}_{(X,V)} = \mathbb{P}_X \otimes \mathbb{P}_V$ with Radon-Nikodym deri. $Z_t(X, Y)$.

Since $\mathbb{P}_{Y|X} = Z_t(X, Y) \mathbb{P}_V$ by Girsanov: $Y_{0..t} \mid X_{0..t}$ is BM with drift $H(X_t)$.

Then we can compute $\mathbb{E}(I_A(X_t) H(Y_{0..t}))$

$\Rightarrow$ We can use the Kallianpur-Striebel formular to obtain a SPDE for unnormalized conditional dist. $\tilde{\eta}_t(A) := \mathbb{E}_{\bar{X}}(I_A(\bar{X}_t) Z_t(\bar{X}, Y))$:

For $f \in C^\infty(\mathbb{R}^d)$, by Itô's, we have

$d\mathbb{E}_{\bar{X}}(f(\bar{X}_t) Z_t(\bar{X}, Y))$

$= \mathbb{E}_{\bar{X}}(Z_t(\bar{X}, Y)(Lf(\bar{X}_t)dt + f(\bar{X}_t)\langle R^{-1}H(\bar{X}_t), dY_t\rangle))$

$\Rightarrow d\tilde{\eta}_t(f) = \tilde{\eta}_t(Lf)dt + \tilde{\eta}_t(f \cdot H) R^{-1} dY_t$ where L is generator for $\bar{X}_t$. It's called Zakai-equation in weak form.

If $\tilde{\eta}_t(dx) = \tilde{\eta}_t(x)dx$, $\tilde{\eta}_t$ is smooth density

$\Rightarrow \partial_t \tilde{\eta}_t(x) = L^* \tilde{\eta}_t(x)dt + \tilde{\eta}_t(x)H(x)R^{-1}dY_t$.

Rmk: To get SPDE for η_t, we can apply Itô's with $h(x,y) = x/y$ $(y>0)$.

$$\Rightarrow d\eta_t(f) = \eta_t(Lf)dt - (\eta_t(fH) - \eta_t(f)\eta_t(H)) \cdot R^{-1}(dY_t - \eta_t(H)dt)$$

It's called Kushner-Stratonovich eq.

If $\eta_t(dx) = \eta_t(x)dx$, then

$$\partial_t \eta_t(x) = L^* \eta_t(x)dt + (H(x) - \eta_t(H))\eta_t(x)R^{-1} \cdot (dY_t - \eta_t(H)dt)$$

(2) Kalman-Bucy Filter:

For $B(x) = Bx$, $H(x) = Hx$ are linear, $C(x) = C$ const. and $X_0 \sim N(m_0, P_0)$. i.e.

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} B & 0 \\ H & 0 \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix}dt + \begin{pmatrix} C & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} dW_t \\ dV_t \end{pmatrix}$$

Rmk: Note it's a Gaussian process $\Rightarrow \eta_t = N(m_t, P_t)$ is again normal dist. since $(X,Y) \sim N(m,\sigma) \Rightarrow X|Y \sim N(m_Y, \sigma_Y)$

By Itô's, we have Kalman-Bucy filter eq.

for m_t, P_t:

$$dm_t = Bm_t\,dt + P_t H^T R^{-1}(dY_t - Hm_t\,dt)$$

$$\frac{d}{dt}P_t = BP_t + P_t B^T + CC^T - P_t H^T R^{-1} H P_t.$$

Rmk: Sol. m_t of the SDE is $E(X_t \mid Y_{0,t})$.