Linear SPDEs

(1) Itô calculus in Hilbert space:

Def: For H is Hilbert space.

i) $(M_t)_{t\le T}$ is conti. H-valued mart. s.t. $\mathbb{E}\|M_t\|^2$ is unif. bdd.

Rmk: $\exists$ unique $\langle M\rangle_t \in \mathcal{G}_t$. conti. increasing. s.t. $\|M_t\|^2 - \langle M\rangle_t$ is martingale.

ii) $\mathcal{L}_+^1(H) =: \{$ semi-definite self-adjoint linear positive trace class on $H\}$.

$(M_t \otimes M_t)_{t\le T}$ is $\mathcal{L}_+^1(H)$-valued. def by: $\langle (M_t\otimes M_t) h, k\rangle_H = \langle M_t, h\rangle\langle M_t, k\rangle$.

Thm. (Métivier, Pistone)

There exists unique conti. adapted. increasing $\mathcal{L}_+^1(H)$-valued process $\langle\langle M\rangle\rangle_t$ s.t. $M_t \otimes M_t - \langle\langle M\rangle\rangle_t$ is mart.

Besides. $\exists$ unique $\mathcal{L}_+^1(H)$-valued predictable process (Q_s). s.t. $\langle\langle M\rangle\rangle_t = \int_0^t Q_s \, d\langle M\rangle_s$

Prop. $\operatorname{Tr}\langle\langle M\rangle\rangle_t = \langle M\rangle_t$. So: $\operatorname{Tr} Q_t = 1$ a.e.

Pf: $\operatorname{Tr}(M_t \otimes M_t - \langle\langle M\rangle\rangle_t) =$

$\|M_t\|^2 - \operatorname{Tr}(\langle\langle M\rangle\rangle_t)$ is mart.

$\Rightarrow \operatorname{Tr}\langle\langle M\rangle\rangle_t = \int_0^t \operatorname{tr} Q_s \, d\langle M\rangle_s$

$= \langle M\rangle_t$.

Def. (B_t^k) is scalar Bm. $Q \in \mathcal{L}_+^1(H)$.

(e_k) is o.n.b. of H. We call:

$W_t =: \sum_{k\geq 0} B_t^k Q^{\frac{1}{2}} e_k$ H-valued Wiener process.

Rmk: It's easy to see. $\langle W\rangle_t = \operatorname{tr} Q \cdot t$

and $Q_t = Q / \operatorname{tr} Q$.

Thm (Characterization of H-valued Wiener)

For $(W_t)_{t\leq T}$, conti. H-valued mart.

Then (W_t) is Wiener process $\Leftrightarrow$

$\exists c$ and $\exists Q \in \mathcal{L}_+^1(H)$. $Q_t = Q$. $\langle W\rangle_t = ct$.

Def. $(\varphi_t)_{t\leq T}$ is H-valued predictable s.t.

$\int_0^T (Q_t \varphi_t, \varphi_t)_H \, d\langle M\rangle_t < \infty$ a.s. set:

$\int_0^t (\varphi_s, dM_s) =: \lim_{N\to\infty} \sum_1^{\tilde{N}} \left(\int_{\tilde{t}_k}^{\tilde{t}_{k+1}} \varphi_s \, ds, M_{t\wedge \tilde{t}_{k+1}}\right.$

$\left. - M_{t\wedge \tilde{t}_{k+1}}\right) / (\tilde{t}_{k+1} - \tilde{t}_k) \Big)_H$

Rmk: i) $\int_0^t \langle \varphi_s, dM_s \rangle_H$ is $\mathbb{R}^1$-valued mart.

with $\langle \int_0^\cdot \langle \varphi_s, dM_s \rangle_H \rangle_t = \int_0^t \langle Q_s \varphi_s, \varphi_s \rangle_H \, d\langle M \rangle_s$.

ii) When W_s is Wiener process. Then:

$$\int_0^t \langle \varphi_s, dW_s \rangle_H = \lim_{N \to \infty} \sum_1^N \int_0^t \langle \varphi_s, e_k \rangle_H \, dB_s^k$$

Thm. (Itô Formular)

For (X_t), (V_t), (M_t) H-valued process.

s.t. $X_t = X_0 + V_t + M_t$. V_t is BV process

with $V_0 = 0$. M_t is local mart with $M_0 = 0$. $\phi \in C^1(H \to \mathbb{R}^1)$

If ϕ'' exists in Gateau sense. $h \mapsto Tr(\phi''(h) Q)$ is conti. for $\forall Q \in L^1(H)$. Then:

$$\phi(X_t) = \phi(X_0) + \int_0^t \langle \phi'(X_s), dV_s \rangle + \int_0^t \langle \phi'(X_s), dM_s \rangle + \frac{1}{2} \int_0^t Tr(\phi''(X_s) Q_s) \, d\langle M \rangle_s$$

Cor. $\|X_t\|^2 = \|X_0\|^2 + 2\int_0^t \langle X_s, dV_s \rangle + 2\int_0^t \langle X_s, dM_s \rangle + \langle M \rangle_t$.

Pf: Since $\phi(x) = \|x\|^2$. $tr Q_t = 1$. $\forall t$.

(2) Definitions and Regularity:

Consider $x \in B$. separable Banach space. L is generator of $S_t \in C_0$ on B. W_t is cylindrical Wiener on K. Hilbert. $Q: K \to B$. BLD.

We want to solve:

$$dX = LX\,dt + Q\,dW_t. \quad X(0) = X_0. \quad (*)$$

Rmk: QW_t may not be B-valued Wiener. and X may not $\in D(L)$. So we can't simply integrate both sides.

① Def: i) X is weak solution for $(*)$. if:

(a). $\forall t > 0$. $\int_0^t \|X(t)\|^2 dt < \infty$. a.s.

(b). $\langle l, X(t)\rangle = \int_0^t \langle L^* l, X(s)\rangle ds + \int_0^t \langle Q^* l, dW_s\rangle$

for $\forall l \in D(L^*)$.

ii) X is mild solution for $(*)$ if:

$$X(t) = S(t)X_0 + \int_0^t S(t-s)\,Q\,dW_s.$$

Rmk: i) Note for $f \in C(\mathbb{R}_+; D(L))$

$$X(t) = S(t)X_0 + \int_0^t S(t-s) f(s)\,ds$$

solves $\partial_t X = LX + f$.

We replace f by $Q\,dW$. above.

ii) By Markov of W_t, we have:

$$X(t) = S(t-s)X(s) + \int_s^t S(t-r)\, dW_r.$$

<u>prop.</u> Mild solution $\underset{\longleftarrow}{\xrightarrow{\text{a.s.-integrable}}}$ Weak Solution.

<u>Pf</u>: WLOG. set $X_0 = 0$. (Or set $X_t - S_t X_0$)

($\Rightarrow$) $\forall\, l \in D(L^*)$, by def:

$$\int_0^t \langle L^* l, X_s \rangle ds = \int_0^t \int_0^s \langle L^* l, S(s-r)\, dW_r \rangle ds$$

$$= \int_0^t \langle \int_r^t S^*(s-r) L^* l\, ds, dW_r \rangle$$

Note $S_t^* \in C_0$ on $B^+ = \overline{D(L^*)}$.

$$\Rightarrow \text{RHS} = \int_0^t \langle S^*(t-r) l, dW_r \rangle - \int_0^t \langle l, dW_r \rangle$$

$$= \langle l, X(t) \rangle - \int_0^t \langle l, dW_r \rangle.$$

Extend $l \in D(L^*)$ to $D(L^*)$ by the weak*-dense of $D(L^*)$ in $D(L^*)$.

($\Leftarrow$) Set $\Sigma =: C([0,t], D(L^*)) \cap C^1([0,t], B^+)$

First, prove:

$$\langle f(t), X_t \rangle = \int_0^t \langle f'(s) + L^* f(s), X_s \rangle ds + \int_0^t \langle f(s), dW_s \rangle, \quad \forall f \in \Sigma.$$

$\overset{\text{dense}}{\Longleftarrow}$ prove $f = \varphi(t) l$, $\varphi \in C^1([0,1], \mathbb{R})$, $l \in D(L^*)$

Note $\frac{d}{dt} \langle \varphi_t l, X_t \rangle = \varphi_t' \langle l, X_t \rangle + \varphi_t \langle l, X_t' \rangle$

$= \cdots$. it holds.

So: Set $f(s) = S^*(t-s) l$, $l \in D(L^*)$.

$\Rightarrow \langle l, X_t \rangle = \int_0^t \langle l, S(t-s) Q\, dW_s \rangle, \ \forall l \in D(L^*)$.

Note B^* is separating points in B.

So $X_t = \int_0^t S(t-s) Q\, dW_s$ a.s.

② Time and space regularity:

Thm. H, K are separable Hilbert. $L \in C_0$ on H. $Q: K \to H$ is BLO. W_t is cylindrical Wiener on K. If $\| S_t Q \|_{HS} < \infty$, $\forall t$, a.s. and $\int_0^1 t^{-2\alpha} \| S_t Q \|_{HS}^2 < \infty$ for some $\alpha \in (0, \frac{1}{2})$. Then: solution of $(*)$ is conti. a.s.

Pf: Note: $\| S(t+s) Q \|_{HS} \leq \| S(s) \| \, \| S(t) Q \|_{HS}$

$\Rightarrow \int_0^T t^{-2\alpha} \| S(t) Q \|_{HS}^2 dt < \infty, \ \forall T < \infty$.

From id.: $\int_s^t (t-r)^{\alpha-1} (r-s)^{-\alpha} dr = 1/c_\alpha$.

$\Rightarrow X(t) = S_t X_0 + c_\alpha \int_0^t \int_s^t (\cdots)(\cdots) S(t-s) \cdots dW_s$

$\overset{\text{Fubini}}{=} S_t X_0 + c_\alpha \int_0^t S(t-r) \eta(r) (t-r)^{\alpha-1} dr$.

Where $\eta(r) = \int_0^r S(r-s) (r-s)^{-\alpha} Q\, dW_s$.

So $\mathbb{E} \| \eta(t) \|^2 \overset{\text{unif.}}{<} \infty$. Fix $T \in \mathbb{R}^+$

with Fernique's: $\mathbb{E} \int_0^T \| \eta(t) \|^p \leq C_p$.

$\Rightarrow \eta \in L^p$. use $\tilde{\eta}$ conti replace η.

It's easy to check $X(t)$ is a.s. conti!

Thm. Set $B = H$ Hilbert. L generates analytic semigroup. and denote H_α the corresp'd interpolation space. If $\exists\, \alpha \geq 0$. st. $Q: K \to H_\alpha$ BLO. and $\beta \in [0, \frac{1}{2}+\alpha]$. st. $\|(-L)^{-\beta}\|_{HS} < \infty$. Then. $x \in H_\gamma$. $\forall\, \gamma < \gamma_0 = \frac{1}{2}+\alpha-\beta$

Rmk: It tells us in which interpolation space we can find the solution.

Cor. If in addition L is self-adjoint. Then: $x \in H_{\gamma_0} = H_{\frac{1}{2}+\alpha-\beta}$.

Pf: WLOG. set $0 \in \rho(L)$. (For using some esti.)

Note:
$$\int_0^T \|(-L)^\gamma S_t Q\|_{HS}^2\, dt \leq$$
$$C \int_0^T \|(-L)^\gamma S_t (-L)^{-\alpha}\|_{HS}^2\, dt$$
$$\lesssim \int_0^T \|(-L)^{-\beta}\|_{HS} \; \|(-L)^{\beta+\gamma-\alpha} S_t\|\, dt$$
$$\lesssim \int_0^T 1 \vee t^{\alpha-\gamma-\beta}\, dt < \infty.$$

Thm. (Time regularity)

In the setting of Thm above. For $\forall$ fixed $\gamma < \gamma_0$. $\Rightarrow$ X_t is a.s. δ-Hölder conti in H_γ. for $\forall\, \delta < \frac{1}{2} \wedge (\gamma_0 - \gamma)$.

Pf: Check Kolomogrov's conti. Criterin.

$$\mathbb{E}\, \|X_t - X_s\|_\gamma^2 \leq C\,|t-s|^{1 \wedge 2(\tilde{\gamma}-\gamma)}.\ \forall\, \tilde{\gamma} < \gamma_0$$

(3) Long time Behavior:

· The long time behavior of solution may depend on choice of x_0, L, Q, and μ.

<u>Def</u>: i) $\mathcal{B}_b(B) =: \{ f : B \to \mathbb{R}^1 \mid f$ is bdd and Borel measurable $\}$.

ii) (P_t). BLO on $B_b(B)$. defined by

$$P_t \varphi(x) =: \mathbb{E}(\varphi(S_t x + \int_0^t S(t-s) Q \, dW_s)).$$

<u>Rmk</u>: i) $P_t : C_b(B) \to C_b(B)$.

easy to check.

ii) $P_t \mathbb{1} = \mathbb{1}$. $P_t \varphi \geq 0$ if $\varphi \geq 0$.

$P_t I_A$ is p.m. on B. $\forall x$.

iii) $P_t \circ P_s = P_{t+s}$ is semigroup.

iii) $P_t(x, \cdot)$ is law of $S_t x + \int_0^t S(t-s) Q \, dW_s$

<u>Rmk</u>: $P_t \varphi(x) = \int_B \varphi(\eta) P_t(x, d\eta)$.

extend to $\mu \in \mathcal{P}(B) =: \{$ finite TV and Borel measurables $\}$:

$$P_t^* \mu(A) =: \int_B P_t(x, A) \mu(dx).$$

iv) Borel p.m. μ on B is invariant for $(*)$ if $P_t^* \mu = \mu$. $\forall t \geq 0$.

Lemma $\widehat{P_t \mu}(x) = \hat{\mu}(S_t^* x)\, e^{-\frac{1}{2}\langle x, Q_t x\rangle}$ for $\mu \in P(B)$.

where $Q_t =: \int_0^t S_t Q Q^* S_t^* \, dt$.

Pf: $LHS = \int_B \mathbb{E}\left(e^{i\langle h - \int_0^t S(t-s) Q \, dW_s, \, S_t^* x\rangle}\right) \mu(dh)$

follows from transform of variable.

Thm. For $B = H$. Hilbert space. Q_t define as above.

Then: i) invariant measure μ for $(*)$ exists.

ii) $\sup_{t>0} \operatorname{tr} Q_t < \infty$.

iii) $\exists$ positive definite trace class op.

$Q_\infty : H \to H$. st. $2\operatorname{Re}\langle Q_\infty L^* x, x\rangle + \|Q^* x\|^2 = 0$. $\forall x \in D(L^*)$.

i). ii). iii) are equivalent.

Furthermore. $\forall$ invariant measure has form. $\nu * \mu_\infty$. Where ν is measure on H invar. under op. $S(t)$ and $\mu_\infty \sim N(0, Q_\infty)$.

Rmk: invar. under $P_t \overset{\text{corspd.}}{\longleftrightarrow}$ invar. under S_t

Pf: i) $\Rightarrow$ ii) By Lemma. with $|\hat{\mu}(x)| \le 1$.

$\Rightarrow \langle x, Q_t x\rangle \le -2\log|\hat{\mu}(x)|$.

estimate $|1-\hat{\mu}(x)|$.

ii) $\Rightarrow$ iii) $\langle x, Q_\infty x\rangle = \int_0^\infty \langle S_r Q Q^* S_r^* x, x\rangle dr = \int_0^t + \int_t^\infty$

$= \int_0^t \| Q^* S_r^* x\|^2 dr + \langle S_t^* x, Q_\infty S_t^* x\rangle.$

for $\forall x \in D(L^*)$.

Take derivative at $t=0$. and real part.

iii) $\Rightarrow$ i) Set $F_x : t \mapsto \langle Q_\infty S_t^* x, S_t^* x\rangle$. $x \in D(L^*)$.

Note $F_x(t) - F_x(0) \overset{iii)}{=} -\int_0^t \| Q^* S_t^* x\|^2 dt$.

$\partial_t F_x(t) \overset{iii)}{=} 2\mathrm{Re}\langle Q_\infty L^* S_t x, S_t^* x\rangle$.

$\underset{\text{dense}}{\overset{D(L^*)}{\Rightarrow}} Q_\infty = S_t Q_\infty S_t^* + Q_t$.

Combined with the Lemma above.

$\mu_\infty \sim N(0, Q_\infty)$ is invariant w.r.t (*)

Besides. $\nu * \mu_\infty$ is invar. under the cond.

Conversely. Set $\mu_t = S_t^* \mu$. $\Rightarrow \hat{\mu}_t(x) = \hat{\mu}(S_t^* x)$.

By Lemma. and invar. of μ.

$\widehat{P_t \mu}(x) = \hat{\mu}_t(x)\, e^{-\frac{1}{2}\langle Q_t x, x\rangle} = \hat{\mu}(x)$. So:

$\hat{\mu}_t(x) \xrightarrow{t\to\infty} \varphi(x)$. $\exists \varphi$. unif. on bdd set.

$\Rightarrow$ prove : $\exists \nu$. p.m. s.t. $\varphi = \hat{\nu}$.

So: We only need to prove : (μ_t) is tight.

<u>Rmk</u>: If $X_0 \sim \nu$. invar. under S_t. Then

$X_t = S_t X_0 + \int_0^t S(t-s) Q\, dW_s$

$\xrightarrow{t\to\infty} X_\infty \sim \nu * \mu_\infty$. if Q_∞ exists.

Prop. If $\lim_{t\to\infty} \|S_t x\| = 0 \quad \forall x \in H$. Then $(*)$ has at most one invar. measure.

Besides, if such invar. measure $\tilde{\mu}$ exists, then $P_t \nu \xrightarrow{v} \tilde{\mu}$ for $\forall$ p.m. ν on H.

Pf: i) δ_0 is the unique measure invar. on S_t:

If $S_t^* \nu = \nu$. Then: $\forall \varphi \in C_B(H, \mathbb{R})$.

$$\int \varphi \, d\nu = \lim_{t\to\infty} \int \varphi(S_t x) \, d\nu(x) = \varphi(0).$$

ii) Set $\mu_t \sim N(0, Q_t)$. p.m. on H.

Since $Q_t \uparrow Q_\infty$. we have $\mu_t \xrightarrow{v} \mu'$.

where $\mu' \sim N(0, Q_\infty)$.

So: $P_t^* \nu = (S_t^* \nu) * \mu_t \xrightarrow{v} \delta_0 * \mu'$

Rmk: i) $\lim_{t\to\infty} \|S_t x\| = 0$ isn't sufficient for the existence of invar. measure.

ii) If invar. measure $\tilde{\mu}$ exists, but $\overline{\lim_{t\to\infty}} \|S_t x\| > 0$. for some x. Then:

$P_t^* \delta_x \xrightarrow{v} \tilde{\mu}$ won't hold!

Prop. (Speed of convergence).

$\dim H < \infty$. If $E V(L) \subset [\operatorname{Re} z < 0]$. $|Q_\infty| \neq 0$.

Then $\exists\, T > 0$. s.t. $P_t^* \delta_x$ has smooth density $p_{t,x}$ w.r.t. Lebesgue measure. $\forall t > T$.

Besides. μ_∞ has smooth density $p_\infty(\cdot)$. also.

and $\exists c>0$. s.t. for $\forall \lambda > 0$. we have:

$$\lim_{t\to\infty} e^{ct} \sup_{y\in\mathbb{R}^n} e^{\lambda|y|} |p_\infty(y) - p_{t,x}(y)| = 0$$

<u>Pf</u>: Note $\|S_t x\| \xrightarrow{t\to\infty} 0$. Since $S_t =: e^{tL}$.

Then use explicit form of density.

<u>Rmk</u>: Generally. $\|\mu_t - \mu_\infty\| \to 0$ may not hold.

<u>Def</u>: Given $V: B \to \mathbb{R}^+$. weight function

i) $\|\varphi\|_V =: \sup_B |\varphi(x)| / (1+V(x))$

ii) $\|\mu - \nu\|_{TV,V} =: \sup \{ \int \varphi d\mu - \int \varphi d\nu \mid \|\varphi\|_V = 1 \}$.

<u>Rmk</u>: Note $V>0$. So: $\|\mu - \nu\|_{TV} \le \|\mu - \nu\|_{TV,V}$

we consider a stronger convergence:

whether $\|P_t^* \nu - \tilde{\mu}\|_{TV,V} \to 0$.

<u>Thm</u>. (Harris)

P_t is Markov semigroup on polish space X.

If $\exists T_0 > 0$. and $V: X \to \mathbb{R}^+$. s.t.

i) $\exists \gamma < 1$. $K>0$. s.t. $P_{T_0} V(x) \le \gamma V(x) + K$. $\forall x$.

ii) $\forall K' > 0$. $\exists \delta > 0$. s.t. $V(x) + V(y) \le K' \Rightarrow \|P_{T_0}^* \delta_x - P_{T_0}^* \delta_y\|_{TV} \le 2 - \delta$.

Then: $\exists T > 0$. $c<1$. s.t. $\|P_T^* \mu - P_T^* \nu\|_{TV,V} \le c \|\mu - \nu\|_{TV,V}$

Thm. If $(*)$ has solution in $B = \mathcal{H}$. Hilbert space

and $\exists T > 0$. s.t. $\|S_T\| < 1$. and $S_T : \mathcal{H} \to R(Q_T^{\frac{1}{2}})$.

Then. $(*)$ has a unique invar. measure $\tilde{\mu}$.

and $\exists \gamma > 0$. s.t. $\| P_t^* \nu - \tilde{\mu} \|_{TV} \le C_\nu e^{-\gamma t}$.

for $\forall t \ge T$. ν p.m. on $\mathcal{H}$. with finite 2^{nd} Moment.

Pf: 1) Note $\|S_t\| < 1$. $\forall t \ge T$.

$R(S_t) \downarrow$. $R(Q_t^{\frac{1}{2}}) \uparrow$.

Also. $R(S_t) \subset R(Q_t^{\frac{1}{2}})$. $\forall t \ge T$.

Next. we will check condition i). ii).

in Harris Thm. when $t = T$.

2) Set $V(x) = \|x\|$. $\mu_t \sim \mathcal{N}(0, Q_t)$.

$\Rightarrow P_T V(x) \le \|S_T x\| + \int \|x\| \, d\mu_T$.

3) Set $\|h\|_T =: \inf \{ \|x\| \mid h = Q_T^{\frac{1}{2}} x \}$. $\forall h \in R(Q_T^{\frac{1}{2}})$

By $C < T$. $\|S_T x\|_T \le C \|x\|_T$.

$\| P_T^* \delta_x - P_T^* \delta_y \|_{TV} \underset{\text{Formula.}}{\overset{\text{CM}}{=}} \| \mathcal{N}(0,1) - \mathcal{N}(\|S_T x - S_T y\|_T, 1) \|_{TV}$

4) Existence is from Banach's fix point Thm.

Cor. $x \mapsto P_t^* \delta_x$ is conti. w.r.t $\|\cdot\|_{TV}$ if $t \ge T$

Lemma. (Identity $R(Q_T^{\frac{1}{2}})$)

$R(Q_t^{\frac{1}{2}}) = R(A_t)$. Where $A_t : \begin{array}{l} L^2([0,T], \mathcal{K}) \longrightarrow \mathcal{H} \\ h(s) \longmapsto \int_0^t S_r Q h_s \, ds \end{array}$

Pf: Note $Q_t = A_t A_t^*$. $\underset{\text{decop.}}{\overset{\text{Polar}}{\Longrightarrow}}$ $Q_t^{\frac{1}{2}} = A_t J_t$. $\exists J_t$. iso.