Semilinear SPDEs

(1) **Definitions:**

Consider L is generator of $S \subset C_0$ on B separable Banach space. W_t is cylindrical Wiener on K. Separable Hilbert space. $F: D(F) \subset B \to B$. measurable. and $Q: K \to B$ is BLO. Given $(\Omega, \mathbb{P})$. with $\mathcal{F}_t := \sigma(\Omega_s, s \le t)$

$$dX(t) = LX\,dt + F(X)\,dt + Q\,dW_t. \quad X(0) = X_0 \in B. \quad (*)$$

<u>Def</u>: i) $t \mapsto X(t) \in D(F)$. is mild solution to $(*)$.

if. $X_t = S_t X_0 + \int_0^t S(t-s) F(X_s)\,ds + W_L(t)$.

for $\forall t > 0$. holds. a.s. where $W_L(t) :=$

$\int_0^t S(t-s)\,Q\,dW_s$.

It's called local mild solution if $\exists \tau$.

$\mathcal{F}_t$ - stopping time. Only holds. a.s $t < \tau$

ii) A local mild solution (X, τ) is max if

$\forall$ mild solution $(\tilde{X}, \tilde{\tau})$. we have $\tilde{\tau} \le \tau$. a.s.

<u>Rmk</u>: Set $\tilde{L} = L - c$. $\tilde{F} = F + c$. for some $c \in \mathbb{R}^1$.

the solution still stays invariant.

So. WLOG. Set S_t is bdd semigroup.

Thm (Existence and Unique)

If $F: B \to B$ is local Lipschitz. $W_L(t)$ is conti. Then, there exists unique max mild solution (X, τ) to $(*)$. with conti. sample path st. $\lim_{t \uparrow \tau} \|X_t\| = \infty$ a.s. on $[\tau < \infty]$.

Besides. F is global Lipschitz $\Rightarrow$ $\tau = \infty$ a.s.

Pf: It relies on Banach fixed point Thm.

Set $M_{g,T}(u)(t) := \int_0^t S(t-s) F(u(s)) ds + g(t)$

for $g(t) = S_t X_0 + W_L(t) \in C[0,T]$

Assume: $\|S_t\| \le M$. $\forall t \ge 0$. we have:

$$\begin{cases} \sup_{[0,T]} \|M_{g,T}(u)(t) - M_{g,T}(v)(t)\| \le MT \sup_{[0,T]} \|F_{u(t)} - F_{v(t)}\| \\ \sup_{[0,T]} \| M_{g,T}(u)(t) - g(t) \| \le MT \sup_{[0,T]} \|F_{u(t)}\|. \end{cases}$$

choose $T < \sim$ to satisfies Banach fixed point Thm. for $M_{g,T}: C([0,T], B) \to C([0,T], B)$.

Thm. If S_t is analytic Semigroup. and $W_L(t)$ has a.s.-conti path in B_α for some $\alpha \ge 0$. Besides, $\exists\, \gamma \ge 0$. and $\delta \in [0,1)$. st. $\forall \beta \in [0, \gamma]$. F can extend from $B_\beta \to B_{\beta-\delta}$ local Lipschitz and grow at most polynomial. Then: $\exists$ unique max mild solution (X, τ). st. $X \in B_\theta$ $\forall \theta < \theta_* = \alpha \wedge (\gamma + 1 - \delta)$.

Pf: Similar as above. by assumptions:

$$\sup_{[0,T]} \| M_{q,T}(u)(t) - M_{q,T}(v)(t) \| \leq M T^{1-\delta} \sup_{[0,T]} \| F_{u(t)} - F_{v(t)} \|.$$

Let $W_L^a(t) = \int_{at}^{t} S(t-r)\, a\, dW_r$. $a \in [0,1)$.

Next. we will use "Bootstrap" argument.

prove: $\forall \theta \in [0, \theta^*)$. $\exists P_\theta \geq 1$. $\gamma_\theta \geq 0$. and $a \in (0,1)$. $C > 0$. st.

$$\|X_t\|_\theta \leq C t^{-\gamma_\theta} (1 + \sup_{[at,t]} \|X_s\| + \sup_{[0,t]} \|W_L^a(s)\|_\theta)^{P_\theta}.$$

1') It's true for $\theta = 0$. $P_\theta = 1$. $\gamma_\theta = 0$

2') If it's true for $\theta_0 \in [0, \gamma)$.

Then. $\forall \varepsilon \in (0, 1-\delta)$. we have:

$$\|X_t\|_{\theta_0+\varepsilon} \leq C t^{-\varepsilon} \sup_{[at,t]} (1 + \|X_s\|_{\theta_0}^n) + \|W_L^a(t)\|_{\theta_0+\varepsilon}$$

since $X_t = S((1-a)t) X_{at} + \int_{at}^{t} \square + W_L^a(t)$.

with assumptions on F.

$\Rightarrow$ By hypo on θ_0. we have $\theta_0 + \varepsilon$ holds.

Rmk: It relies on the regularity of F from one interpolation space into another.

(2) Sobolev Embedding:

Next. we restrict SPDEs on $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

Def: The fractional Sobolev space $H^s(\mathbb{T}^d)$ for $s \geq 0$ is subspace of $L^2(\mathbb{T}^d)$:

$$\{ u \in L^2(\mathbb{T}^d) \mid \sum_{\mathbb{Z}^d} (1+|k|^2)^s |\hat{u}_k|^2 =: \|u\|_{H^s} < \infty \}.$$

Rmk: i) $s = 0$. $H^s = L^2$; $s < 0$. we regard H^s as closure of L^2 under $\|\cdot\|_{H^s}$.

ii) $H^s = D((1-\Delta)^{\frac{s}{2}})$ and $\|u\|_{H^s} = \|(1-\Delta)^{\frac{s}{2}} u\|_{L^2}$. by Fourier.

iii) If $L = \Delta$. $\mathcal{H} = H^s$. Then: the interpolation space $\mathcal{H}_\alpha = H^{s+2\alpha}$ connecting with fractional Sobolev space.

prop. A is positive definite self-adjoint on $\mathcal{H}$ separable Hilbert. For $\alpha \in [0,1]$. we have: $\|A^\alpha u\| \leq \|Au\|^\alpha \|u\|^{1-\alpha}$. $\forall u \in D(A^\alpha)$.

Pf: By Riesz repre. and Hölder's.

Cor. $\forall t > s$. $r \in [s,t]$. $\Rightarrow \|u\|_{H^r}^{t-s} \leq \|u\|_{H^t}^{r-s} \|u\|_{H^s}^{t-r}$

Pf: Set $\mathcal{H} = H^s$. $A = (1-\Delta)^{\frac{t-s}{2}}$. $\alpha = \frac{r-s}{t-s}$

Lemma. $\forall s > \frac{d}{2}$. $H^s(\mathbb{T}^d) \subset L^\infty(\mathbb{T}^d) \cap C^\alpha(\mathbb{T}^d)$. $\forall \alpha < s - \frac{d}{2}$.

and $\exists C > 0$. $\|u\|_{L^\infty} \leq C \cdot \|u\|_{H^s}$

Pf: By Lemma. easy to check.

Thm. (Sobolev embedding)

$p \in [2,\infty]$. we have: $\forall\, s > \frac{d}{2} - \frac{d}{p}$. $H^s(\mathbb{T}^d) \subset L^p(\mathbb{T}^d)$. and $\exists\, C>0$. s.t. $\|u\|_{L^p} \le C\|u\|_{H^s}$.

Pf: By Lemma. only prove: $p \in [2,\infty)$.

The idea is to divide $u = \sum \hat{u}_k e^{ikx}$ into blocks: $u_{(n)} := \sum_{2^n+1}^{2^{n+1}} \hat{u}_k e^{ikx}$ and estimate each part.

Set $s' = d/2 + \varepsilon > \frac{d}{2}$. So: $\|u^{(n)}\|_{L^\infty} \le C\|u^{(n)}\|_{H^{s'}}$

Note $\|u^{(n)}\|_{L^p}^p \le \|u^{(n)}\|_{L^2}^2 \|u^{(n)}\|_{L^\infty}^{p-2}$ with

$$\begin{cases} \|u^{(n)}\|_{L^2} \le 2^{-ns}\|u\|_{H^s} \\ \|u^{(n)}\|_{L^\infty} \le C\, 2^{n(s'-s)}\|u\|_{H^s} \end{cases}$$

$$\overset{\text{set}}{\underset{\varepsilon\to 0^+}{\Longrightarrow}} \|u^{(n)}\|_{L^p} \le C\cdot 2^{-n\delta}\|u\|_{H^s}. \text{ for some } \delta > 0.$$

Rmk: If $p \in [2,\infty)$ $\Rightarrow$ $H^{\frac{d}{2}-\frac{d}{p}} \subset L^p$ also holds.

Thm. $r, s, t \in \mathbb{R}^{>0}$. s.t. $t < (s\wedge r)\wedge(s+r-\frac{d}{2})$.

Then: $u \in H^r$. $v \in H^s$ $\Rightarrow$ $uv \in H^t$.

Pf: Set $w = uv$. $\hat{w}^{(m,n)} = u^{(m)} v^{(n)}$.

$\Rightarrow$ $\hat{w}^{(m,n)}_k = 0$ if $k < 2^{1+m\vee n}$.

So: $\|W^{(m,n)}\|_{H^t} \leq C\, 2^{t(m\vee n)} \|u^{(m,n)}\|_{L^2}$

$\overset{\text{Hölder}}{\leq} C\, 2^{t(m\vee n)} \|u^{(m)}\|_p \|v^{(n)}\|_q$

for $1/2 = 1/p + 1/q$. and $r > t + \frac{d}{2} - \frac{d}{p}$.

$s > \frac{d}{2} - \frac{d}{q}$. with the Thm above:

$$\begin{cases} \|u^{(m)}\|_p \leq C \|u^{(m)}\|_{H^{r-t-\varepsilon}} \leq C\, 2^{-m(t+\varepsilon)} \|u^{(m)}\|_{H^r} \\ \|v^{(n)}\|_q \leq C \|v^{(n)}\|_{H^{s-\varepsilon}} \leq C\, 2^{-n\varepsilon} \|v^{(n)}\|_{H^s}. \end{cases}$$

$\Rightarrow \|W^{(m,n)}\|_{H^t} \leq C\, 2^{-\varepsilon(m+n)} \|u^{(m)}\|_{H^r} \|v^{(n)}\|_{H^s}$.

(3) Examples:

① Reaction - Diffusion Eq.

$\partial_t u(x,t) = \Delta u\, dt + f(u)\, dt + Q\, dW_t$ is SPDE modelling the reaction's evolution in a gel. where $u(t,x) \in \mathbb{R}^d$. $x \in D$. is density of reaction at time t. and position x.

Thm. If W_Q is conti. and $\exists\, V \in C^2(\mathbb{R}^d; \mathbb{R}^+)$ convex. s.t. $\lim\limits_{|x|\to\infty} V(x) = \infty$. and $R > 0$. $\exists\, C$. $\langle \nabla V(x), f(x+y) \rangle \leq C\, V(x)$. $\forall x \in \mathbb{R}^d$ and $|y| \leq R$. Then. it has global solution.

(2) Consider :

$$\partial_t u(t,x) = \frac{1}{2} \sum_{ij} \frac{\partial}{\partial x_j} (a_{ij}(t,x)\, \partial_{x_i} u) + \sum_i b_i(t,x) \cdot$$

$$\partial_{x_i} u(t,x) + f(t,x,u) + \sum_k (\sum_i g_{ki}(t,x)$$

$$\cdot \partial_{x_i} u(t,x) + h_k(t,x,u))\, dW_k(t)$$

$u(0,x) = u_0(x)$. satisfies :

i) $\exists\, \sigma > 0$. $a - \sum_k g_k^T g_k \geq \sigma I$. (Posi-definite)

ii) $r \mapsto f(t,x,r)$ is conti.

iii) $r f(t,x,r) + \sum_k |h_k(t,x,r)|^2 \leq c(1+|r|^2)$

iv) $[f(x,x,r) - f(t,x,r')](r-r') + \sum |h_k(t,x,r) - h_k(\cdots r')|^2$

$\leq \lambda |r-r'|^2$. $\exists\, \lambda > 0$.

Rmk: The equation has unique conti and square-integrable solution.

Prop. If $u(0,x) \geq 0$. $f(t,x,0) \geq 0$. $h_k(t,x,0) = 0$ $\forall k$.

Then: $u(t,x) \geq 0$ $\forall t \geq 0$. $x \in \mathbb{R}^d$.