SPDEs driven by Space-time Noise.

(1) <u>Restriction:</u>

① Consider $du(t,x) = \frac{1}{2}\Delta u(t,x)dt + W(t,x)dt$.

$$u(0,x) = u_0(x). \quad x \in \mathbb{R}^d. \quad t \geq 0. \quad (*)$$

where $\mathbb{E}(W(h)W(k)) = \langle h,k\rangle$. $\forall h,k \in L^2$.

Denote $p(t,x) = e^{-|x|^2/2t} / (2\pi t)^{\frac{d}{2}}$.

Then solution of $(*)$ is:

$$u(t,x) = u_0 * p_t(x) + \int_0^t \int_{\mathbb{R}^d} p(t-s,x-y)\, W(ds,dy).$$

But note that $Var(u) = \int_0^t \int_{\mathbb{R}^d} p^2(t-s,x-y)\,ds\,dy$

$$= \frac{1}{2^d \pi^{\frac{d}{2}}} \int_0^t 1/(t-s)^{\frac{d}{2}} < \infty$$

iff $d=1$. So for $d \geq 2$. We set:

$$\langle u,\varphi\rangle = \langle u_0 * p(t,\cdot)(x),\varphi\rangle + \int_0^t \int \left(\int \varphi(x)\, p(t-s,x-y)\,dx \right) dW$$

for $\forall \varphi \in C_c^\infty(\mathbb{R}^d)$.

Note $\int \varphi(x)\, p(t-s,x-y)\,dx = \mathbb{E}_y(\varphi(B_{t-s}))$

$$\leq \|\varphi\|_\infty \mathbb{E}(I_{\{|B_{t-s}| \geq |y|-r\}}) \leq \|\varphi\|_\infty \frac{\mathbb{E}|B_{t-s}|^p}{(|y|-r)^p}.$$

for r. s.t. $\operatorname{supp}\varphi \subset \overline{B}(0,r)$ and $p > d/2$.

So: we have $\mathrm{Var}\langle u(t),\varphi\rangle < \infty$. well-def.

③ When we consider:

$$du(t,x) = \frac{1}{2}\Delta u\,dt + f(u)\,dt + g(u)\,W(t,x)\,dt.$$

$$u(0,x) = u_0(x) \quad . \quad t \geq 0. \quad x \in \mathbb{R}^d.$$

The general form of ①. It will be less regular. So we only restrict on $d=1$.

(2) Existence and Unique:

Consider SPDE with homo-Dirichlet condition:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x,u) + g(t,x,u)\,W(t,x). \quad t \geq 0,\ 0 \leq x \leq 1.$$

$$u(t,0) = u(t,1). \quad u(0,x) = u_0(x). \quad (*')$$

Def: i) $(W(A)) := (\int_A W(ds,dx))$ is random field of center Gaussian. s.t.

$\mathbb{E}(W(A)W(B)) = \mathcal{L}(A\cap B)$. $\mathcal{L}$ is Lebesgue.

ii) $\mathcal{F}_t := \sigma(W(A).\ A \in \mathcal{B}_{[0,t]\times[0,1]})$.

$\mathcal{P} := \sigma((s,t]\times A.\ A \in \mathcal{F}_s)$. predictable.

iii) For $\psi : \mathbb{R}_+ \times [0,1] \times \Omega \to \mathbb{R}^1$. $\in \mathcal{P}\otimes\mathcal{B}_{[0,1]}$.

and $\int_0^t \int_0^1 \psi^2 < \infty$. $\forall t$. a.s. Set:

$$\int_0^t \int_0^1 \varphi(s,x)\, W(ds,dx) := \lim_{n\to\infty} \sum\sum \langle \varphi, I_{A_{i,j}^n}\rangle\, W(A_{i,j}^n \cap \square)$$

where $\square = [0,t]\times[0,1]$, $A_{i,j}^n = [\frac{i}{n},\frac{i+1}{n}]\times[\frac{j}{n},\frac{j+1}{n}]$.

Rmk: $\mathbb{E}\left(\left(\int_0^t\int_0^1 \varphi\, dW\right)^2\right) \overset{iso-}{=} \mathbb{E}\int_0^t\int_0^1 \varphi^2\, dx\, ds$.

iii) $u(t,x)$ is weak solution of $(*')$ if:

$$\langle u(t),\varphi\rangle_{L^2[0,1]} = \langle u_0,\varphi\rangle + \int_0^t \langle u(s),\varphi''\rangle ds + \int_0^t \langle f(s,u(s)),\varphi\rangle ds + \int_0^t \langle g(s,u(s)),\varphi(s)\, W(ds)\rangle, \quad \forall \varphi \in C_c^\infty(0,1)$$

iv) $u(t,x)$ is mild solution of $(*')$ if:

$$u(t,x) = \int_0^1 p(t,x,y)\, u_0(y)\, dy + \int_0^t\int_0^1 p(t-s,x,y)\, f(s,y,u)\, dy\, ds + \int_0^t\int_0^1 p(t-s,x,y)\, g(s,y,u)\, W(ds,dy), \quad \forall\, t\geq 0,\ 0\leq x\leq 1.$$

where $p(t,x,y)$ solves $\begin{cases} \partial_t u = \partial_{xx} u \\ u(t,0) = u(t,1) = 0. \end{cases}$

Rmk: i) $p(t,x,y)$ is like semigroup of A.

ii) $p(t,x,y) = (4\pi t)^{-\frac{1}{2}} \sum_{\mathbb{Z}} \left[e^{-(2n+y-x)^2/4t} - e^{-(2n+y+x)^2/4t} \right]$

and satisfies: $\forall T>0$, $\exists\, C_T > 0$ s.t.

$$|p(t,x,y)| \leq C_T\, e^{-|x-y|^2/4t}/\sqrt{t}, \quad \forall\, t \leq T.$$

prop. If i) $\int_0^t \int_0^1 f^2 + g^2 \, ds\, dx < \infty$. $\forall t \geq 0$

ii) $\exists \delta$. local. bdd. $|\Delta_{r,r'}^{\text{dis}} f(s,x,\cdot) + \Delta_{r,r'}^{\text{dis}} g(s,x,\cdot)| \leq \delta(r)$

Then: $u \in C^2$. $P \otimes B_{(0,1)}$ is weak solution for

$(*')$ $\iff$ u is mild solution for $(*')$.

Pf: $(\Rightarrow)$. $\forall \phi \in C^{1,2}(\mathbb{R}_+ \times (0,1)) \cap C(\mathbb{R}_+ \times [0,1])$.

s.t. $\phi(t,1) = \phi(t,0) = 0$. can be

approxi. by $\sum_1^n \lambda_i(t) \varphi_i(x)$.

So: $\langle u(t), \phi(t,\cdot) \rangle = \cdots$ also holds.

Set $\phi(t,x) = \int_0^1 p(s-t, y, x) \varphi(y)\, dy \hat{=} p(t-s, \varphi, x)$

$\Rightarrow$ $\langle u(t), \varphi \rangle = \cdots$. Let $\varphi \to \delta_x$.

$(\Leftarrow)$. Set $t_i = it/n$. $\Delta t = t/n$.

$\langle u(t), \varphi \rangle - \langle u_0, \varphi \rangle =$

$\sum_0^{n-1} \langle u(t_{i+1}), \varphi \rangle - \langle u(t_i), p(\Delta t, \varphi, \cdot) \rangle +$

$\langle u(t_i), p(\Delta t, \varphi, \cdot) \rangle - \langle u(t_i), \varphi \rangle$.

Insert: $\langle u(t), \varphi \rangle = \langle u(s), p(t-s, \varphi, \cdot) \rangle + \cdots$

into the equation above. Set $n \to \infty$.

the conclusion follows from conti. of u.

Thm (Unique and Existence)

Under the condition above. If $|\Delta_{r,r'}^{\text{dis}} g(s,x,\cdot)$

$+ \Delta_{r,r'}^{\text{dis}} f(s,x,\cdot)| \leq K|r-r'|$. and $u_0 \in C_0[0,1]$.

Then. $\exists$ unique conti. solution $u \in \mathcal{P} \otimes \mathcal{B}_{[0,1]}$ of $(*')$. s.t. $\sup_{[0,1]\times[0,T]} \mathbb{E}(|u(t,x)|^p) < \infty$. $\forall p \geq 1$.

Pf: i) Uniqueness: By Gronwall inequi.

ii) Existence: By Picard iteration procedure.

Cor. The solution u has a a.s. $1/4 - \varepsilon$ Hölder conti modification. $\forall \varepsilon > 0$.

Pf: Check Kolmogorov Lemma.

Cor. (Positivity of solution)

If $u_0(x) \geq 0$. $f(t,x,0) \geq 0$. $g(t,x,0) = 0$

Then: $u(t,x) \geq 0$. $\mathbb{P}$-a.s.

(3) SPDEs and Super BMs:

consider $\partial_t u(t,x) = \frac{1}{2} \partial_{xx} u(t,x) + u^{\gamma} \dot{W}(t,x)$

$u(0,x) = u_0(x)$. $x \in \mathbb{R}^1$. $t \geq 0$ $(\tilde{*})$

where $u_0(x) \geq 0$. So: we have $u(t,x) \geq 0$

$\gamma = 1$: It's trivial

$\gamma > 1$: $r \mapsto r^{\gamma}$ is Lipschitz. $\Rightarrow$ exist unique solution

Next, we consider the case $\gamma < 1$.

① $\frac{1}{2} < \gamma < 1$:

Thm. If $u_0 \in C(\mathbb{R}^1, \mathbb{R}^+)$. s.t. $\sup_{\mathbb{R}} e^{p|x|} u(x) < \infty$ for $\forall p > 0$. Then: $\forall \phi \in D(\Delta)$. $\exists$ unique law m of $u(t,x)$. s.t.

$$Z_t(\phi) := \langle u_t, \phi \rangle - \langle u_0, \phi \rangle - \frac{1}{2}\int_0^t \langle u_s, \Delta\phi \rangle ds$$

is conti. mart. with associated increasing process: $\langle Z(\phi) \rangle_t = \int_0^t \langle u^{2\gamma}, \phi^2 \rangle ds$

Rmk: i) It means the mart problem for $(\widehat{*})$ has unique solution.

ii) No uniqueness result is known when $\gamma < \frac{1}{2}$.

② $\gamma = \frac{1}{2}$:

Def: i) M_d denote the space of finite measures on $\mathbb{R}^d$. $C_{ct}^d \triangleq C_c^2(\mathbb{R}^d, \mathbb{R}^+)$.

$\langle \, , \, \rangle$ is pairing of measure and function $\in C_{ct}^d$.

ii) Super Brownian motion is Markov process $(X_t)_{t \geq 0}$ taking values in

M_d and $t \mapsto \langle X_t, \varphi \rangle$ is right-conti.

for $\forall \varphi \in C_{c^+}^{d}$. characterized by:

$$\mathbb{E}_M (e^{\langle X_t, \varphi \rangle}) = e^{\langle M, V_t(\varphi) \rangle} . \quad \forall \varphi \in C_{c^+}^{d}$$

where $M \in M_d$. $V_t(\varphi)$ is solution of:

$$\begin{cases} \partial_t V = \frac{1}{2} (AV - V^2) \\ V(0) = \varphi . \end{cases} \qquad V_t : \mathbb{R}^{\geq 0} \to C_{c^+}^{d}$$

<u>Rmk:</u> i) By Markov property of X_t

$\Rightarrow$ V_t is semigroup.

Besides. $\mathbb{E}_M (e^{\langle X_t, V_{T-t}(\varphi) \rangle} \mid \mathcal{F}_s)$

$$\overset{\text{Markov}}{=} \mathbb{E}_{X_s} (\square)$$

$$\overset{\text{semigroup}}{=} e^{\langle X_s, V_{T-s}(\varphi) \rangle}$$

$\Rightarrow (e^{\langle X_t, V_{T-t}(\varphi) \rangle})_{t \leq T}$ is mart.

ii) Set $F(X_t) = f \langle X_t, \varphi \rangle$. where

$f : \mathbb{R}^1 \to \mathbb{R}^1$. $\varphi \in C_{c^+}^{d}$

Apply Taylor expansion on $f(\langle X_t, \varphi \rangle)$

We have $\mathbb{E}_M (F(X_t) - F(M)) / t$

$$\overset{t \to 0}{\longrightarrow} \mathcal{G} F(M). \text{ generator of } X_t.$$

$$\mathcal{G} F(M) = \frac{1}{2} f'(\langle M, \varphi \rangle) \langle M, A\varphi \rangle + \frac{1}{2} f''(\langle M, \varphi \rangle) \langle M, \varphi^2 \rangle$$

prop. $\forall \varphi \in C_{ct}^{\wedge}$. $M_t^\varphi = \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle$

$- \frac{1}{2} \int_0^t \langle X_s, \Delta\varphi \rangle ds$ is conti. mart.

with $\langle M^\varphi \rangle_t = \int_0^t \langle X_s, \varphi^2 \rangle ds$.

Pf: Note $f(\langle X_t, \varphi \rangle) - f(\langle \mu, \varphi \rangle) - \int_0^t \mathcal{G} f(\langle X_s, \varphi \rangle)$

is a mart.

First. set $f(x) = x$. we have.

$M_t^\varphi = \langle X_t, \varphi \rangle - \langle \mu, \varphi \rangle - \frac{1}{2} \int_0^t \langle X_s, \Delta\varphi \rangle ds$

is mart.

Then. set $f(x) = x^2$. we have

another mart $N_t^\varphi = \langle X_t, \varphi \rangle^2 - \cdots$

Apply Itô formular on $\phi(M_t^\varphi)$. $\phi = x^2$.

$\Rightarrow$ we obtain $\langle M^\varphi \rangle_t$ by its charac.

Thm. i) $d \geq 2$. $X_t \perp \mathcal{L}$. a.s. $\mathcal{L}$ is Lebesgue

$d = 1$. $X_t \ll \mathcal{L}$. a.s. for $\forall t \geq 0$.

ii) Set $u(t, \cdot)$ is density of X_t. Then.

$\exists$ Gaussian random measure $W(s, x)$ on

$\mathbb{R}^+ \times \mathbb{R}^1$. s.t. $M_t^\varphi = \int_0^t \int_{\mathbb{R}^1} u^{\frac{1}{2}} \varphi(x) W(ds, dx)$.

Cor. $u(t, x)$ is weakly positive solution of

$\partial_t u = \frac{1}{2} \partial_{xx} u + u^{\frac{1}{2}} W(t, x)$. $u(0, x) = u_0(x) \geq 0$.

Construction :

It's an approximation by branching process.

1) At $T=0$. N particles i.i.d. locate in $\mathbb{R}^d$ with law μ.

2) At $T=k/N$. $k\geq 1$. each particles die with prob $=1/2$ and give birth to 2 descedents with prob $=1/2$.

3) On $[(k-1)/N, k/N]$. living particles move as i.i.d. Bm.

Denote: Set $N(t)$ is number of living particles at $T=t$. Y_t^i is position of i^{th} particle at $T=t$.

Thm. $X_t^N := \frac{1}{N}\sum_{i=1}^{N(t)} \delta_{Y_t^i} \xrightarrow{N\to\infty} X_t$. $\sim$ Super Bm. with initial law μ.

Cor. The extinction time τ of X_t $< \infty$. a.s.

Pf: Note $\mathbb{P}(\sup_{1\leq i\leq N} T_i \leq t) = \prod_{i}^{N} \mathbb{P}(T_i\leq t)$

$\sim (1-c/Nt)^N \xrightarrow{N\to\infty} e^{-c/t}$.

where T_i is extinction time of i^{th} particle and we use result from Kol.

So: $\mathbb{P}(\tau > t) \sim 1 - e^{-c/t} \xrightarrow{t\to\infty} 0$.

<u>Lemma.</u> $\forall t \geq 0$. $\varphi \in C_{cr}^{\lambda}$. We have:

$$\mathbb{E}_m (e^{-\int_0^t \langle X_t, \varphi \rangle dt}) = e^{-\langle m, u_t(\varphi) \rangle}.$$

where $u_t(\varphi)$ is positive solution of

$$\begin{cases} \partial_t u = \frac{1}{2}(\Delta u - u^2) + \varphi \\ u(0) = 0. \end{cases}$$

<u>Pf:</u> By approximation:

$$\mathbb{E}_m (e^{-\int_0^t \langle X_t, \varphi \rangle dt}) = \lim_{n\to\infty} \mathbb{E}_m (e^{-\sum_i^n \langle X_{t_i}, \Delta t \cdot \varphi \rangle})$$

<u>Thm.</u> (Compact supp. property)

If $m \in M_\lambda$. st. $\text{supp}(m) \subset B(0, R_0)$. Then. $\forall R > R_0$

$$\mathbb{P}_m (X_t (B(0,R)^c) = 0. \forall t \geq 0) = e^{-\langle m, u(R^{-1} \cdot) \rangle / R^2}.$$

where u is positive solution of

$$\begin{cases} \Delta u = u^2. & \forall |x| < 1. \\ u \to \infty. & x \to \pm 1. \end{cases}$$

<u>pf:</u> Approxi. $I_{B(0,R)^c}$ by $\varphi_n \in C_c^\infty(0, R')$.

and next. We will exploit the fact: $t \mapsto \langle X_t, \varphi \rangle$ is right-conti.

$$LHS \overset{\text{right}}{\underset{\text{conti}}{=}} \mathbb{P}_\mu \left(\int_0^\infty X_t(B(0,R)^c)\,dt = 0 \right)$$

$$= \lim_{\theta \to \infty} \mathbb{E}_\mu \left(e^{-\theta \int_0^\infty X_t(B(0,R)^c)\,dt} \right)$$

$$= \lim_{\theta \to \infty} \lim_{T \to \infty} \lim_{n \to \infty} \mathbb{E}_\mu \left(e^{-\int_0^T \langle X_t, \varphi_n \cdot \theta \rangle\,dt} \right)$$

Apply the Lemma above.

Cor. Under the conditions above.

$$\mathbb{P}_\mu \left(\bigcup_{t \geq 0} \operatorname{supp}(X_t) \text{ is bdd} \right) = 1$$

Pf: $LHS \geq \mathbb{P}_\mu \left(\bigcap_{r > R_0} \{ X_t(B(0,r)^c) = 0, \ \forall t \geq 0 \} \right)$

$$= \lim_{r \to \infty} e^{-\langle \mu, u(r^{-1}\cdot) \rangle / r^2}$$

$$\geq \lim_{r \to \infty} e^{-|\mu| \cdot \|u\|_\infty / r^2} = 1.$$

Rmk: It holds for $\forall \gamma < 1$ as well.