Examples

Dean - Kawasaki equation :

(B_t^i) indept of BMs. $f \in C_B^2(\mathbb{R}^1)$. Denote

$\eta_t^N(f) := \frac{1}{N}\sum_1^N f(B_t^i + X_0) = \int f \, d(\frac{1}{N}\sum_1^N \delta_{B_t^i + X_0})$

$\xrightarrow{a.s.} \mathbb{E}[f(B_t + X_0)]$ by SLLN.

Recall for $p(t,x)$ density of $B_t + X_0$ satisfies:
$$\partial_t p(t,x) = \frac{1}{2}\partial_{xx} p(t,x) \quad \text{(heat eq.)}$$
$$p(t,x)dx \xrightarrow{t \to 0} \delta_{X_0}(dx)$$

Rmk: (B_t^i) can be seen as microscopic behavior. (particle, molecular)

Next, we investigate fluctuation of η_t^N:

Set $\eta_t(dx) = p(t,x)dx$. By above: $\eta_t^N \to \eta_t$

$d\eta_t^N(f) = d(\frac{1}{N}\sum_1^N f(B_t^i + X_0)) \overset{Itô}{=} dM_t^N(f) + \eta_t^N(\frac{1}{2}\partial_{xx}f)dt$

where $dM_t^N(f) = \frac{1}{N}\sum_1^N \partial_x f(B_t^i + X_0)\, dB_t^i$

Let $\Psi_t^N(f) := \sqrt{N}(\eta_t^N(f) - \eta_t(f))$. We have:

$d\Psi_t^N(f) = \sqrt{N}\, dM_t^N(f) + \Psi_t^N(\frac{1}{2}\partial_{xx}f)dt.$

Note $\mathbb{E}(M_t^N(f)^2) = \frac{1}{N}\mathbb{E}(\int_0^t (f'(B_s+X_0))^2 ds)$

$$\langle M_t^N(f)\rangle_t = N^{-2}\sum_1^N \int_0^t f'(B_s^i+X_0)^2 ds$$

$$= O(N^{-1}).$$

So: $M_t^N(f) \to 0$ a.s. Return to $d\eta_t^N = \square$:

$$d\eta_t^N(f) = dM_t^N(f) + \eta_t^N(\tfrac{1}{2}\partial_{xx} f)dt$$

$$\overset{N\to\infty}{\simeq} \eta_t^N(\tfrac{1}{2}\partial_{xx} f)dt$$

If density of $\eta_t^N(\cdot)$ exists:

$$\eta_t^N(f) \simeq \eta_0^N(f) + \int_0^t \eta_s^N(\tfrac{1}{2}\partial_{xx} f) ds$$

$$= \eta_0^N(f) + \int_0^t \int \tfrac{1}{2} f''(x)\eta_s^N(x)dx\,ds$$

$$= \eta_0^N(f) + \int_0^t \int f(x)\tfrac{1}{2}(\eta_s^N(x))'' dx\,ds$$

So: $d\eta_t^N(f) = \frac{1}{2}\partial_{xx}\eta_t^N(f)dt$ (heat eq.)

Besides notice $M_t^N(f) \overset{\sim}{=} N^{-\frac{1}{2}}\int_0^t (\eta_s^N((f')^2))^{\frac{1}{2}} dB_s$

(in sense of 1^{st}, 2^{nd} moments equal.)

Also, we have its limit dist. as:

<u>Lem.</u> $\sqrt{N} M_t^N(f) \xrightarrow{d} \int_0^t (\eta_s((f')^2))^{\frac{1}{2}} dB_s$

$$\sim N(0, \int_0^t \eta_s((f')^2) ds)$$

<u>Pf:</u> $\langle N^{\frac{1}{2}} M_t^N(f)\rangle_t \xrightarrow{SLLN} \int_0^t \eta_s((f')^2) ds$

It leads to SPDE:

$d\eta^N_t(f) = (\eta^N_t((f')^2)/N)^{\frac{1}{2}} dB_t + \eta_t(\frac{1}{2}f'')dt$

Denote space-time white noise $\xi(t,x)$, i.e.

$\mathbb{E}(\xi(t,x)\xi(s,y)) = \delta(t-s)\,\delta(x-y)$

Also, $\frac{1}{N}\mathbb{E}(\int_0^t \eta^N_s((f')^2)ds) =$

$\frac{1}{N}\mathbb{E}(\int_0^t \int_{\mathbb{R}^1} (f')^2(x)\,\eta^N_s(x)\,dx\,ds) =$

$\frac{1}{N}\mathbb{E}((\int_0^t \int_{\mathbb{R}^1} f'(x)\,(\eta^N_s(x))^{\frac{1}{2}}\, d\xi(s,x))^2)$

$\Rightarrow \sqrt{\eta^N_t((f')^2)/N}\; dB_t \cong \frac{1}{\sqrt{N}}\int_{\mathbb{R}^1} f'(x)\sqrt{\eta^N_s(x)}\, d\xi(s,x)$

It leads to Dean–Kawasaki SPDE:

$\partial_t \eta^N_t(x) = \frac{1}{2}\partial_{xx}\eta^N_t(x) + \frac{1}{\sqrt{N}}\partial_x(\sqrt{\eta^N_t(x)}\,\xi(t,x))$

Similarly, let $N\to\infty$ in $\partial_t \Psi^N_t(x) = \square$

$\Rightarrow \partial_t \Psi_t = \frac{1}{2}\partial_{xx}\Psi_t + \partial_x(\sqrt{\eta}\, d\xi(t,x))$

<u>Rmk</u>: It has structure $dX_t = AX_t dt +$

$B\,dW_t$: $A = \frac{1}{2}\partial_{xx}$, $B = N^{-\frac{1}{2}}\partial_x$ and

$\sqrt{\eta^N_s(x)}\, d\xi(s,x) \sim N(0, \eta^N_s(x))$