Sto-Ana on Hilbert

(1) Gaussian measure:

Def: Measure μ on Hilbert space $(H, \langle\cdot,\cdot\rangle, \mathcal{B}_H)$ is Gaussian measure if $\forall h \in H$. $l_h : g \mapsto \langle g, h\rangle$ is normally dist. i.e.

$$\int_H e^{it\, l_h(g)} d\mu(g) = e^{it\, m(h) - \frac{1}{2} t^2 \sigma^2(h)}$$ for

some $m(h) \in \mathbb{R}^1$, $\sigma(h) \in \mathbb{R}^+$.

Rmk: By Riesz. l_h goes over $\mathcal{L}(H, \mathbb{R}^1)$.

In the following we assume H is separable

Thm. μ is Gaussian on H $\Leftrightarrow$ $\forall h \in H$.

$$\int_H e^{i\langle h, g\rangle} \mu(dg) = e^{i\langle m, h\rangle - \frac{1}{2}\langle Qh, h\rangle}$$

where a) $m \in H$ mean vector.

b) Cov. operator $Q \in \mathcal{L}(H)$ sym. positive-definite and of finite trace. i.e. $\operatorname{Tr}(Q) = \sum \langle Qe_k, e_k\rangle < \infty$ for any o.n.b. (e_k)

Rmk: It describes structure of ch.f. for μ. We write $\mu \sim N(m, Q)$.

Cr. i) $\int_H \langle x, h\rangle \, d\mu(x) = \mathbb{E}_\mu(l_h) = \langle m, h\rangle$

ii) $\int_H (\langle x, h_1\rangle - \langle m, h_1\rangle)(\langle x, h_2\rangle - \langle m, h_2\rangle) \, d\mu(x)$

$= \mathrm{Cov}_\mu(l_{h_1}, l_{h_2})$

$= \langle Q h_1, h_2\rangle. \ \forall h_1, h_2 \in H.$

iii) $\int_H \|x - m\|^2 d\mu = tr(Q).$

Pf: i) let $h = th$. Apply $\partial/\partial t|_{t=0}$.

ii) Prove $\int_H (\langle x, h\rangle - \langle m, h\rangle)^2 d\mu = \langle Qh, h\rangle$ as i).

Then apply parallel id.

iii) Note $\|x - m\|^2 = \sum \langle x - m, e_i\rangle^2$ by Parseval id.

Pf: $(\Leftarrow)$ is trivial by Def. For $(\Rightarrow)$:

i) $m(g+h) = \mathbb{E}_\mu(l_{g+h})$

$= \mathbb{E}_\mu(l_g + l_h) = m(g) + m(h).$

$\Rightarrow m(\cdot)$ is linear

And set $H_n := \{h \in H : \int |l_h(x)| d\mu \leq n\}$

H_n is closed by Fatou's Lem.

With $H = \bigcup H_n$. $\Rightarrow \exists n_0$ st. $\text{int} H_{n_0} \neq \emptyset$ from

Baire Thm $\Rightarrow \exists h_0 \in H$. $B_\varepsilon(h_0) \subset H_{n_0}$.

So for $h \in H$. $\|h\| \leq 1$. we have:

$$|m(h)| = |\int l_h(x) d\mu|$$

$$\leq \frac{2}{\varepsilon} \int |l_{h_0 + \frac{\varepsilon}{2} h}| d\mu + \int |l_{h_0}| d\mu \leq \frac{4n_0}{\varepsilon} < \infty.$$

So $m(\cdot) \in H^*$ $\overset{\text{Riesz}}{\Rightarrow} \exists m \in H$. st. $\langle m, h \rangle = m(h)$

Similarly, we can show $Q \in \mathcal{L}(H \times H, \mathbb{R})$.

ii) Next, we prove: $\text{tr}(Q) < \infty$.

<u>Lem.</u> For $\mu \sim N(0, Q)$ Gaussian measure on H. $\Rightarrow 1 + \text{tr}(Q) \leq 1 / \left(\int e^{-\|x\|^2/2} d\mu(x)\right)^2$

<u>Rmk:</u> In the case H is Hilbert. then CM space of H is $H_\mu := R(Q^{\frac{1}{2}}) = R(Q)$ with product $\langle k, h \rangle_{H_\mu} := \langle Q^{-\frac{1}{2}} k, Q^{-\frac{1}{2}} h \rangle_H$. So: the Lem. ensures H_μ is well-def and $H_\mu \overset{\text{cpt}}{\hookrightarrow} H \Rightarrow H_\mu$ is Sep. Hilb.

Pf: 1) For $\dim H = n < \infty$. If $(\sigma_i^2)_1^n$ is eigenvalues of Q. then:

$$\int e^{-\|x\|^2/2} d\mu(x) = \prod_1^n (1+\sigma_i^2)^{-\frac{1}{2}}.$$

$$1 + tr(Q) = 1 + \Sigma \sigma_i^2 \leq \prod_1^n (1+\sigma_i^2) = \square^{-2}$$

2) For $\dim H = \infty$. Set $\Pi_n : H \to \mathbb{R}^n$ s.t. $\Pi_n(h) = (\langle h, e_1 \rangle, \cdots \langle h, e_n \rangle)$. $(e_k)_1^\infty$ is o.n.b. $\mu_n = N(0, Q_n) := \Pi_n^* \mu$.

$$1 + tr(Q_n) = 1 + \sum_1^n \sigma_i^2 \leq \int_H e^{-\frac{1}{2}\sum_1^n \langle h, e_i \rangle^2} d\mu(h).$$

Let $n \to \infty$. apply MCT.

e.g. (Wiener measure)

$(\beta(t))$ is 1-dim SBM on $(\Omega, \mathbb{P}, \mathcal{F})$.

Note $\beta_\cdot : w \in \Omega \mapsto (t \mapsto \beta_t(w)) \in L^2[0,T]$

$\Rightarrow \mu(A) \stackrel{\Delta}{=} \mathbb{P}(\beta(\cdot) \in A)$ is Wiener measure on $L^2[0,T]$ with $m = 0$. $\langle Qg, h \rangle = \int_0^T \int_0^T g(s) h(t) \, t \wedge s \, dt ds = \langle (-\Delta)^{-1} g, h \rangle$. where $\Delta g = g''$ with $g(0) = g'(T) = 0$

Pf: $\mathbb{E}(dh) = \int_\Omega \langle h, \beta \rangle_{L^2} d\mathbb{P}(\beta)$

$$= \int_0^T h(t) \int_\Omega \beta(t) d\mathbb{P} dt = 0.$$

$\mathrm{Cov}(l_h, l_g) = \int_\Omega \int_0^T h(s)\beta(s) \int_0^T g(t)\beta(t)$

$= \int_0^T \int_0^T g(t)h(s) \int_\Omega \beta(s)\beta(t)$

$= \int_0^T \int_0^T g(t)h(s)\, t\wedge s \, dt\, ds.$

And note $Qg(t) = \int_0^T g(s)\, s\wedge t\, ds$

$= \int_0^t g(s)\, s\, ds + t\int_t^T g\, ds$

$\Rightarrow (Qg)''(t) = -g(t)$. i.e. $-\Delta Q = id_{L^2}$

with $Qg(0) = (Qg)'(T) = 0$.

Note $tr(Q) < \infty \Rightarrow Q$ is cpt. We have:

<u>Prop.</u> (Canonical Representation)

If Q is positive semidefinite sym. BLO. s.t. $tr(Q) < \infty$. $(e_k)_1^\infty$ is o.n.b. & they satisfy $Qe_k = \lambda_k e_k$. eigenvectors.

For $m \in H$. Then: $X \sim N(m, Q) \iff X = \sum_1^\infty \sqrt{\lambda_k}\beta_k e_k + m$. $(\beta_k) \overset{i.i.d.}{\sim} N(0,1)$ in sense of convergence a.s. and L^p. $\forall p \geq 1$.

<u>Rmk</u>: It offers us a method to simulate Gaussian r.v. on H.

Pf: ($\Leftarrow$) prove: $\sum_1^m \sqrt{\lambda_k}\beta_k e_k + m =: S_m \to X$ in H

Note $\|S_n - S_m\|_H^2 = \sum_{m+1}^n \lambda_k \beta_k^2$.

So: $\mathbb{E}(\|S_n - S_m\|_H^2) = \sum_{m+1}^n \lambda_k \to 0$

Prove: $X \sim N(m, Q)$.

Note $\mathbb{E}(e^{i\langle h, X\rangle}) = e^{i\langle m, h\rangle} \prod_1^\infty f_k$

$f_k = \mathbb{E}(e^{i\langle h, e_k\rangle \sqrt{\lambda_k}\beta_k}) = e^{-\frac{1}{2}\lambda_k \langle e_k, h\rangle^2}$

So: LHS $= e^{i\langle m, h\rangle} e^{-\frac{1}{2}\sum \langle Q e_k, h\rangle \langle e_k, h\rangle}$

$= e^{i\langle m, h\rangle} e^{-\frac{1}{2}\langle Qh, h\rangle}$

from Parseval id.

($\Rightarrow$) $\exists \beta_k \overset{i.i.d}{\sim} N(0,1)$. st. $\langle X - m, e_k\rangle = \sqrt{\lambda_k}\beta_k$. Since $\langle X - m, e_k\rangle \sim N(0, \lambda_k)$

$\sum_1^m \sqrt{\lambda_k}\beta_k e_k + m \overset{H}{\to} X$ similarly as above

And easy to prove $\sum_1^m \overset{L^p}{\to} \square$ & a.s.

(2) Wiener Process:

Fix $(U, \langle\cdot\rangle_U)$ is separable Hilbert and Q is positive semidefinite sym. BLO. st. $tr(Q) < \infty$

Def: U-valued SP $(W_t)_{[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is Q-Wiener process if

i) $W(0) = 0$. ii) $t \mapsto W(t)$ is conti. a.s.

iii) $(W(t_i) - W(t_{i-1})) \sim N(0, (t_i - t_{i-1})Q)$. indept.

W_t is Q-Wiener process w.r.t. $(\mathcal{F}_t)$

if i) $W_t \in \mathcal{F}_t$ ii) $t \mapsto W_t$ is conti. a.s.

iii) $W_t - W_s \sim N(0, (t-s)Q)$ indept of $\mathcal{F}_s$.

Rmk: $\mathcal{F}_t$ can be chosen as canonical filtration $\mathcal{F}_t^W = \bigcap_{s>t} \sigma(W(r), r \leq s)$.

prop. (Canonical represent for Q-Wiener)

(e_k) is o.n.b. eigenvector of Q. Then:

$(W(t))_{t \in [0,T]}$ is Q-Wiener $\iff$ $W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k$. in sense of convergence on $C([0,T], U)$ and $L^2(\Omega; C([0,T], U))$ $\mathbb{P}$-a.s. where λ_k is eigenvalue of e_k.

$(\beta_k(t)) \overset{i.i.d.}{\sim}$ 1-dim BM.

Rmk: It works for L^p. $\forall 1 < p < \infty$. from mart. converge Thm for U-value mart.

Pf: ($\Leftarrow$) Set $W^n(t) = \sum_1^n \sqrt{\lambda_k} \beta_k(t) e_k$.

$\|W^n(t) - W(t)\|_U^2 = \sum_{n+1}^{\infty} \lambda_k \beta_k(t)^2$.

So $\mathbb{E}(\sup_{[0,T]} \| \cdots\cdots \|_U^2) \le \sum_{n+1}^{m} \lambda_k \mathbb{E}(\sup_{[0,T]} |\beta_k(t)|^2)$

$\overset{\text{Doob's}}{\le} C_T \sum_{n+1}^{\infty} \lambda_k \to 0.$

And " $\sum_1^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k$ is Q - Wiener " is easy to check as before

($\Rightarrow$) Set $\beta_k(t) = \langle W(t), e_k \rangle / \sqrt{\lambda_k} \, I_{\{\lambda_k > 0\}}$.

Check $(\beta_k(t)) \overset{\text{i.i.d}}{\sim}$ 1-dim BM and convergence in L^2 follows from above.

For Q is only bdd but not of finite trace. We can still define one kind of Wiener process called cylinder Wiener.

Assume $(U_1, \langle \cdot \rangle_{U_1})$ is another Hilbert space s.t. $\exists J : U \to U_1$. Hilbert - Schmidt embedding. i.e. $\sum_1^{\infty} \| J e_k \|_{U_1}^2 < \infty$ for (e_k) o.n.b. of U and $U \subset U_1$ dense.

<u>RMK:</u> (U_1, J) always exists. <u>e.g.</u> $(\alpha_k) \in \ell^2$. $\alpha_k \neq 0$. $\forall k$

Let $U_1 = U$. Set $J(u) = \sum_1^{\infty} \alpha_k \langle u, e_k \rangle e_k$

<u>Def:</u> $(W(t))_{[0,T]}$ is U_1 - valued cylinder Q - Wiener if it's Gaussian and satisfy

$\mathbb{E}(\langle h, W_s\rangle_{u_1} \langle k, W_t\rangle_{u_1}) = s\wedge t \langle QJ^*k, J^*h\rangle_u$

for $\forall h, k \in u_1$

<u>Rmk</u>: i) It's not U-valued process:

Otherwise, for $h, k \in R(J^*) \subset U$ and let $Q = I$ for convience.

$\mathbb{E}(\langle h, W_s\rangle_u \langle k, W_s\rangle_u) = (s\wedge t)\langle h, k\rangle_u$

$\Rightarrow$ It means it's I-Wiener. But it's impossible.

ii) For $Q = I$. We have represent:

$W(t) = \sum_1^{\infty} \beta_k(t) e_k$ by prop. above.

But $(e_k) \overset{\text{o.n.b}}{\subset} U$ not converge in U

while $\mathbb{E}(\|J(W(t))\|_{u_1}^2) = t\|J\|_{L_2^0}$

So we just think $J(W(t)) \in U_1$ as $W(t)$. Which's reinteprет of the Rmk i)

(3) <u>Stochastic Integration:</u>

① <u>Mart. in Banach:</u>

Fix $(E, \|\cdot\|)$ is $\mathbb{R}'$-Banach. And $(\Omega, \mathcal{A}, \mu)$ is finite measure space. Next, we define integral $\int_\Omega f d\mu$ for $f: \Omega \to E$.

For step func. $f = \sum_1^n X_k I_{A_k}$, $X_k \in E$. Set $\int_\Omega f d\mu := \sum_1^n X_k \mu(A_k) \in E$.

<u>Lem.</u> (Bochner's inequi.)

$\|\int_\Omega f d\mu\|_E \leq \int_\Omega \|f\| d\mu$. $\forall f = \sum_1^n X_k I_{A_k}$, $X_k \in E$.

<u>Def:</u> For $f: \Omega \to E$.

i) f is strongly measurable if $f = \lim_n f_n$ μ-a.e. for (f_n) step func's.

<u>Rmk:</u> f is weakly measurable if $\forall$ $l \in E^*$, $l(f)$ is $\mathcal{A}$-$\mathcal{B}_{\mathbb{R}'}$ measura.

Note $f \in \mathcal{A}$-$\mathcal{B}_E \Rightarrow f$ is weakly measura. but $\not\Rightarrow$ strongly measura. if $\dim E = +\infty$.

ii) $\mathcal{L}'(\mu) := \{ f$ is strongly measur. $: \int_\Omega \|f\| d\mu < \infty \}$

iv) f is separably valued if $f(\Omega)$ is separable subset of E.

Thm. (Pettis)

f is strongly measura. $\Leftrightarrow$ weakly measura. and separable valued.

By using Bochner's inequi. we can extend integration on step func's to $\mathcal{L}^1(\mu)$.

Prop. $\forall f \in \mathcal{L}^1(\mu; E)$.

i) (Bochner inequi.) $\| \int_\Omega f \, d\mu \|_E \le \int_\Omega \|f\|_E \, d\mu$.

ii) (linear) $L \in \mathcal{L}(E, F)$. F is Banach.

Then: $L(\int_\Omega f \, d\mu) = \int_\Omega L(f) \, d\mu$.

iii) (Founda. Thm for calculus)

For $f \in C^1([a,b], E)$. Then: $\forall s \le t$,

$f(s) - f(t) = \int_t^s f'(r) \, dr$.

Rmk: For ii), in particular, $L: E \times E \to E: (x_1, x_2) \mapsto \alpha x_1 + \beta x_2$. $\alpha, \beta \in \mathbb{K}^1$.

Next, we assume E is separable.

Thm. (Conditional expectation on E)

$X \in L^1(\Omega; E)$. $\mathcal{A}_0 < \mathcal{A}$ sub-σ-algebra. Then:

$\exists$ $\mathbb{P}$-a.s. unique $X_0 \in L^1(\Omega, \mathcal{A}_0, \mu; E)$. s.t.

$\int_A X_0 d\mathbb{P} = \int_A X d\mathbb{P}$. $\forall A \in \mathcal{A}_0$.

Denote $X_0 := \mathbb{E}(X | \mathcal{A}_0)$. cond. exp.

Pf: First consider $X_n \xrightarrow{L^1} X$ step r.v.'s

to approxi. $\mathbb{E}(X|\mathcal{A}_0)$ by $\mathbb{E}(X_n | \mathcal{A}_0)$

Cor. (Bochner) $\|\mathbb{E}(X|\mathcal{A}_0)\|_E \le \mathbb{E}(\|X\|_E | \mathcal{A}_0)$

Pf: By approxi. from step func.

e.g. $\mathcal{A}_0 = \sigma(A_k,\ k=1,\cdots n)$. A_k disjoint $\Rightarrow$

$\mathbb{E}(X|\mathcal{A}_0) = \sum_{k=1}^n \int_{A_k} X d\mathbb{P} \cdot I_{A_k} / \mu(A_k)$.

Def: E-value SP (M_t) on $(\Omega, \mathcal{A}, \mathbb{P})$ is

$(\mathcal{A}_t)$-mart. if

i) $\mathbb{E}(\|M_t\|) < \infty$. $\forall t \ge 0$. ii) $M_t \in \mathcal{A}_t$. $\forall t$.

iii) $\mathbb{E}(M_t | \mathcal{A}_s) = M_s$. $\forall\ 0 \le s \le t$.

Rmk: (M_t) is E-$\mathcal{A}_t$-mart. $\Leftrightarrow$ $\forall \ell \in E^*$:

$(\ell(M_t))$ is $\mathbb{R}^1$-$\mathcal{A}_t$-mart. if i)

holds in Def.

Rmk: Sub-/Super-mart. can't be generalized at once since Banach space lacks total order "≤". It needs extra structure if $\dim E = +\infty$.

e.g. U-valued Q-Wiener process (W_t) is $\mathcal{L}^2$-conti. $\mathcal{G}_t^B$-mart.

Thm. (Doob's)

For $p > 1$. (M_t) is right-conti. $\mathcal{A}_t$-mart.

Then: $\mathbb{E}(\sup_{[0,T]} \|M_t\|^p) \le (\frac{p}{p-1})^p \mathbb{E}(\|M_T\|^p)$

Cor. $\mathcal{M}_T^p := \{(M_t)_{[0,T]}$ E-valued conti. $(\mathcal{A}_t)$-mart. $\|M\|_{\mathcal{M}_T^p} := \sup_{[0,T]} \mathbb{E}(\|M_t\|_E^p) < \infty\}$ is Banach space.

Rmk: It implies L^p-limit of conti. $(\mathcal{A}_t)$-mart. is still conti. $\mathcal{A}_t$-mart.

② Ito-integral:

Let U, H are separa. $\mathbb{R}$-Hilbert. (W_t) is U-valued Q-Wiener process. $\phi : [0,T] \to (\Omega \to L(U,H))$

Next, we construct $\int_0^t \phi(r) dW_r \in H$.

1°) Consider $\phi(t) = \sum_{1}^{N} \phi_m I_{(t_m, t_{m+1}]}(t)$ where $\phi_m : \Omega \to L(U, H)$, $\phi_m \in \mathcal{F}_m$. Denote set $\mathcal{E}$ is collection of such func's. Set $I : \phi \in \mathcal{E} \mapsto I(\phi) = \int_0^t \phi(s) dW_s := \sum_{1}^{n} \phi_m (W_{t_{m+1} \wedge t} - W_{t_m \wedge t})$

2°) Wiener isometry:

Consider Hilbert-Schmidt op. space $L_2(U, H)$ with $\langle A, B \rangle := \sum \langle Ae_k, Be_k \rangle_H$. (e_k) is o.n.b.

<u>prop.</u> H is separ. $\Rightarrow L_2(U, H)$ is separa.

<u>Pf</u>: Set $(y_k)_k$ is dense set of H.

$\Rightarrow (L_n : u \mapsto \sum_{1}^{n} \langle u, e_k \rangle y_{nk})$ is dense set of $L_2(U, H)$.

Since $\forall L \in L_2(U, H)$, $\exists \|y_{nk} - Le_k\| \leq \varepsilon$

$\Rightarrow \|L - L_N\|^2_{L_2(U,H)} \leq \varepsilon^2 \vee \sum_{n+1}^{\infty} \|Le_k\|^2 \to 0$

<u>Rmk</u>: If $\dim U = \dim H = +\infty$, even U, H are separa. $\not\Rightarrow L(U, H)$ is separa.

Next, we prove: for $\phi \in \mathcal{E}$, we have:

$$\mathbb{E}(\|\int_0^t \phi(s) dW_s\|_H^2) = \mathbb{E}(\int_0^t \|\phi(s) \circ \sqrt{Q}\|^2_{L_2(U,H)} ds)$$

Cor. $\| \int_0^{\cdot} \Phi(s) dW_s \|_{M_T^2}^2 = \mathbb{E}\left(\int_0^T \| \Phi(s) \circ \sqrt{Q} \|_{L_2(U,H)}^2 ds \right)$

Pf: By conditional expectation technique

$$\mathbb{E}\left(\langle \Phi_m(e_k), \Phi_n(e_l) \rangle (\beta_k(t_{m+1}) - \beta_k(t_m)) \cdot (\beta_l(t_{n+1}) - \beta_l(t_n)) \right)$$

$$= \delta_{\{m=n, k=l\}} (t_{m+1} - t_m) \mathbb{E}\left(\| \Phi_m(e_k) \|_H^2 \right).$$

for $\beta_k(t) = \langle W(t), e_k \rangle_U$.

$$\Rightarrow LHS = \sum_{m=1}^{N} \sum_{k=1}^{\infty} \lambda_k \mathbb{E}\left(\| \Phi_m(e_k) \|_H^2 \right) (t_{m+1} - t_m)$$

$$= RHS$$

3) Extension:

Note from 2), we have $I : \mathcal{E} \to M_T^2$ is isometry $\| I(\Phi) \|_{M_T^2} = \| \Phi \|_T := \mathbb{E}\left(\int_0^T \| \Phi \circ \sqrt{Q} \|_{\square}^2 \right)$

So we can extend I to $\bar{I} : \bar{\mathcal{E}} \to M_T^2$

where $\bar{\mathcal{E}} = \bar{\mathcal{E}}^{\| \cdot \|_T} = L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T ; L_2^0)$. St.

$\mathcal{P}_T$ is predictable σ-algebra on $\Omega_T = \Omega \times [0,T]$. $\mathbb{P}_T = \mathbb{P} \otimes dt|_{[0,T]}$. $L_2^0 = \{ T \in \mathcal{L}(U,H) : T \circ \sqrt{Q}$ is Hilbert-Schmidt $\}$.

4) Localization:

It can be extended to $\Phi: \Omega_T \to L(U, H)$ adapted. $\Phi \in \mathcal{P}_T$ & $\mathbb{P}(\int_0^T \|\Phi(s) \circ \sqrt{Q}\|^2_{L_2(U,H)} ds < \infty) = 1$

by set $\tau_n := \inf\{t \geq 0 \mid \int_0^t \|\Phi(s) \circ \sqrt{Q}\|^2_{\square} ds > n\} \wedge T$

$\tau_n \uparrow T$. Consider $\Phi_n(t) = \Phi(t) I\{t \leq \tau_n\} \to \Phi(t)$

<u>properties:</u>

i) Linearity: $\forall L \in L(H, \tilde{H})$.

$$L(\int_0^t \Phi(s) dW_s) = \int_0^t L \circ \Phi(s) dW_s.$$

ii) $f: \Omega_T \to H$. $\mathcal{F}_t$-adapted. conti. Then:

$\int_0^t \langle f(s), \Phi(s) dW_s \rangle_H = \int_0^t \tilde{\Phi}_f(s)(dW_s)$ where

$\tilde{\Phi}_f(s)(u) = \langle f(s), \Phi(s)u \rangle : U \to \mathbb{R}^1$.

iii) BDG inequi.: $\forall p \geq 1$. $\exists c_p > 0$. st. $(\tilde{\Phi} = \Phi \circ Q^{\frac{1}{2}})$

$\mathbb{E}(\sup_{[0,t]} \|\int_0^s \Phi(r) dW_r\|_H^{2p}) \leq c_p \mathbb{E}(\int_0^T \|\tilde{\Phi}(s)\|^2_{L_2^0})^p$.

iv) Quadratic Variation: for $M_t = \int_0^t \Phi(s) dW_s$

$\langle M \rangle_t := \int_0^t \|\tilde{\Phi}(s)\|^2_{L_2^0} ds$ is unique conti. $\uparrow$

$\mathcal{F}_t$-adapted process. st. $\langle M \rangle_0 = 0$ and that

$\|M_t\|_H^2 - \langle M \rangle_t$ is $\mathcal{F}_t$-mart.

<u>Rmk:</u> $\forall (\tau_n)$ partition of $[0, T]$. $|\tau_n| \to 0$.

$\Rightarrow \sum_{\substack{(2^n): \\ t_i \le t}} \| M_{t_{i+1}} - M_{t_i} \|_H^2 \xrightarrow{pr} \langle M \rangle_t$ uniform in t.

And $\forall h \in H$. $\int_0^t \| (J_R \circ \Phi^*(s))(h) \|_H^2 ds$ is also RV for $\langle M_t, h \rangle$ having property as $\langle M \rangle_t$.

V) Regularity: $W^{\alpha,2}([0,T], E) := \{ M \in L^2([0,T], E):$
$\int_0^T \int_0^T \| M_t - M_s \|_E^2 / |t-s|^{1+2\alpha} ds dt < \infty \}$.

prop. $\forall \alpha < 1/2$. $M_t = \int_0^t \Phi(s) dW_s \in W^{\alpha,2}([0,T], H)$
for $\forall \Phi \in L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; L_2^0)$

pf: By Fubini and Itô-isometry:

$$\mathbb{E}\left(\int_0^T \int_0^T \square\right) = \int_0^T \int_0^T \frac{\mathbb{E}\left(\int_{s \wedge t}^{t \vee s} \|\tilde{\Phi}(r)\|_{L_2^0}^2 dr\right)}{|t-s|^{1+2\alpha}}$$

$$\le \mathbb{E}\left(\int_0^T \|\tilde{\Phi}(r)\|_{L_2^0}^2 dr\right) \int_0^T \int_0^T |t-s|^{-(1+2\alpha)} ds dt$$

③ Sto-integral on cylinder Wiener:

We can extend $\int_0^t \Phi(s) dW_s$ for W just U_1-valued cylinder Q-Wiener.

Assume $Q = I$. Note that $Q_1 = J \circ J^* \in L(U_1)$ is sym. positive definite with finite trace

And $W_1(t) = J(W(t)) = \sum_1^\infty \beta_k(t) J(e_k)$ is Q_1-Wiener process and converges in $\mathcal{M}_T^2(U_1)$.

Note $\|\Phi(s)\|^2_{L_2(U_0,H)} = \|\Phi(s)\circ J^{-1}\|^2_{L_2(J(U_0),H)}$

Set $N_W = \{\Phi : \Omega_T \to L_2(U_0,H) \mid \Phi$ is pred. & $\mathbb{P}(\int_0^T \|\Phi(t)\|^2_{L_2(U_0,H)} ds < \infty) = 1\}$

$\Rightarrow \forall \Phi \in N_W$. Set $\int_0^t \Phi(s)\,dW_s = \int_0^t \Phi(s)\circ J^{-1}\,dW_1(s)$