SDE on Hilbert

Next, consider separa. $\mathbb{R}'$-Hilbert space U, H and Q-Wiener (W_t) w.r.t. $(\mathcal{F}_t)$ on U.

Consider semilinear SPDE:

$$\begin{cases} dX_t = (AX_t + B(X_t))dt + C(X_t)\,dW_t \in H. \\ X_0 = \xi \in \mathcal{F}_0,\ \xi \in H. \end{cases} \qquad (*)$$

st. i) $(A, D(A))$ generates C_0-semigroup (T_t) on H.

ii) $B : H \to H$, $\mathcal{B}_H$-measur. $C : H \to L_2(U, H)$ st. $x \mapsto C(x)u$, $H \to H$. conti. for $\forall u \in U$.

<u>Rmk</u>: B, C can depend on time t.

(1) <u>Definitions</u>:

<u>Def</u>: i) Mild sol. to $(*)$ is H-valued pred. process $(X_t)_{[0,T]}$ satisfying: $\mathbb{P}$-a.s.

$$X_t = T_t\xi + \int_0^t T_{t-s}B(X_s)ds + \int_0^t T_{t-s}C(X_s)dW_s$$

<u>Rmk</u>: i) X_t isn't semimart generally because of $\int_0^t T_{t-s}\,\square\, dW_s$.

(While for $r \ge t$, $\int_0^t T_{r-s}\,\square\, dW_s$ is)

ii) X_t can really solve (*). which motivates from variation of const

<u>e.g.</u> for $\xi_t' = A\xi_t + f_t$. we have

$$\xi_t = e^{tA}\xi_0 + \int_0^t e^{(t-s)A} f(s)\,ds$$

Now we replace $e^{tA} = T_t$ and

$$f_t = B(X_t) + C(X_t)\frac{dW_t}{dt} \quad \text{formally.}$$

ii) Strong sol. to (*) is $D(A)$-valued pred. process $(X_t)_{[0,T]}$ satisfying: $\mathbb{P}$-a.s.

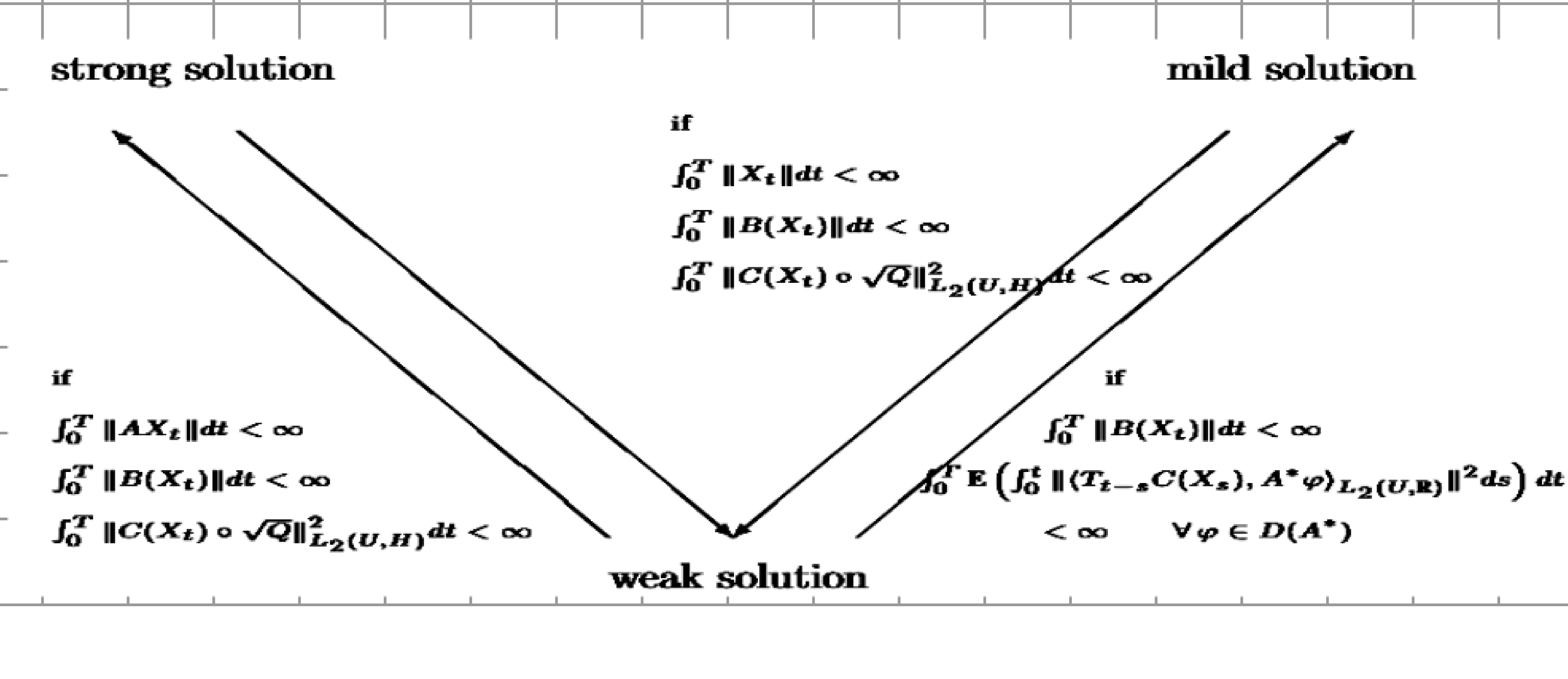
$$X_t = \xi + \int_0^t (AX_s + B(X_s))\,ds + \int_0^t C(X_s)\,dW_s.$$

<u>Rmk</u>: Note X_t should $\in D(A)$ for well-def.

iii) Weak sol. to (*) is H-valued pred. process (X_t) satisfying: $\mathbb{P}$-a.s. $\forall \varphi \in D(A^*)$

$$\langle X_t, \varphi\rangle_H = \langle \xi, \varphi\rangle_H + \int_0^t \langle X_s, A^*\varphi\rangle_H + \langle B(X_s), \varphi\rangle_H\,ds$$
$$+ \int_0^t \langle \varphi, C(X_s)\,dW_s\rangle_H$$

<u>Relation</u>:



(2) Existence and Unique:

Assumpt: i) $\|B(t,x)-B(t,y)\|_H + \|[C(t,x)-C(t,y)]\circ\sqrt{Q}\|_{L_2(H,H)} \leq L\|x-y\|_H$

ii) $\|B(t,x)\|_H^2 + \|C(t,x)\circ\sqrt{Q}\|_{L_2(H,H)}^2 \leq M(1+\|x\|_H^2)$

Thm. Under assumpt i) - ii). Then there exists unique mild sol. to (*) st.

$\mathbb{P}(\int_0^T \|X_s\|_H^2 ds < \infty) = 1$ and $\exists$ conti. modifi.

$\widehat{X}_t$. st. $\mathbb{E}(\sup_{[0,T]} \|X_t\|_H^p) \leq C_{p,T}(1+\mathbb{E}(\|\xi\|_H^p))$. $\forall p\geq 2$

Pf: Apply Banach fixed pt Thm:

For $p\geq 2$. $\mathcal{H}_p := \{Y: \Omega_T \to H$. pred. : $\|Y\|_p^p = \sup_{[0,T]} \mathbb{E}(\|Y(t)\|_H^p) < \infty\}$.

$\Rightarrow (\mathcal{H}_p, \|\cdot\|_p)$ is Banach space.

Set $\mathcal{K}(Y)(t) = T_t\xi + \int_0^t T_{t-s}(B(s,Y_s))ds + \int_0^t T_{t-s}(C(s,Y_s))dW_s$

$\triangleq T_t\xi + \mathcal{K}_1(Y)(t) + \mathcal{K}_2(Y)(t)$.

WLOG. $\mathbb{E}(\|\xi\|_H^p) < \infty$. by localization.

To estimate $\mathbb{E}(\|\mathcal{K}(Y_1)(t) - \mathcal{K}(Y_2)(t)\|^p)$

Recall $\sup_{[0,T]} \|T_t\| \leq C_T$. using Hölder's and BDG inequi. with assumpt i). ii)

$\Rightarrow \mathbb{E}(\|\mathcal{K}(Y_1)(t) - \mathcal{K}(Y_2)(t)\|_H^p)$

$$\leq C_p(T^p + T^{\frac{p+1}{2}}) \|Y_1 - Y_2\|_p^p$$

Choose T small $\Rightarrow \mathcal{K}$ is contraction.

As for conti. modification:

<u>prop.</u> For $p>1$. $\phi \in L^{2p}(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; L_2^0)$. $\exists C_T > 0$

$$\mathbb{E}(\sup_{[0,T]} \|\int_0^t T_{t-s}\phi(s)dW_s\|_H^{2p}) \leq C_T \mathbb{E}(\int_0^T \|\phi(s)\circ\sqrt{Q}\|_{L_2(H,H)}^{2p} ds)$$

<u>Pf:</u> Use factorization method:

We want to split $T_{t-s} = T_{t-r}\circ T_{r-s}$ backward and forward and estimate them separately.

i) Note $\int_s^t (t-r)^{\alpha-1}(r-s)^{-\alpha} dr = \pi/\sin(\pi\alpha)$ $=: C_\alpha^{-1}$. for $\forall \alpha \in (0,\frac{1}{2})$. $\forall s<t$.

(Set $u = r-s/t-s$. $LHS = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)}$)

$$\int_0^t T_{t-s}\phi(s)dW_s = C_\alpha\int_0^t (t-r)^{\alpha-1} T_{t-r}(y(r))dr$$

where $y(r) = \int_0^r (r-s)^{-\alpha} T_{r-s}(\phi(s))\,dW_s$

where we use Sto-Fubini Thm and linearity of T_t.

2) For $\forall p > \frac{1}{2\alpha} > 1$. We use Hölder inequi.

$\|\int_0^t T_{t-s}\phi(s)\,dW_s\|_H^{2p}$

$\leq C_\alpha^{2p} \left(\int_0^t (t-r)^{\frac{2p}{2p-1}(\alpha-1)} dr\right)^{2p-1} \int_0^t \|T_{t-s}(y(r))\|_H^{2p} dr$

$\leq C_{\alpha,p,T} \int_0^T \|T_{t-s}(y(r))\|_H^{2p} dr \leq C_{\alpha,p,T}\int_0^T \|y(r)\|_H^{2p} dr$

By BDG inequi. and Hölder's:

$\mathbb{E}(\|y(r)\|_H^{2p}) \leq C_p \mathbb{E}[(\int_0^r (r-s)^{-2\alpha} \|\phi(s)\circ\sqrt{Q}\|_{L_2}^2 ds)^p]$

$\leq C_p (\int_0^r (r-s)^{-2\alpha})^{p-1}.$

$\mathbb{E}(\int_0^r (r-s)^{-2\alpha} \|\phi(s)\circ\sqrt{Q}\|_{L_2}^{2p} ds)$

So: $\int_0^T \mathbb{E}(\|y(r)\|_H^{2p}) dr \leq$

$C_{\alpha,p,T} \sup_{r\leq T} \int_0^r (r-s)^{-2\alpha} ds\ \mathbb{E}(\int_0^T \|\phi(s)\circ\sqrt{Q}\|_{L_2}^{2p} ds)$

Combined with above, we have:

$\mathbb{E}(\sup_{[0,T]} \|\int_0^t T_{t-s}\phi(s)\,dW_s\|_H^{2p})$

$\leq C_{p,T}\,\mathbb{E}(\int_0^T \|\phi(s)\circ\sqrt{Q}\|_{L_2}^{2p} ds)$

Next, we will show $F: \eta \in L^{2p}([0,T], H) \mapsto F(\eta)$

$= (t \mapsto \int_0^t (t-r)^{\alpha-1} T_{t-r}\eta(r)\,dr) \Rightarrow F(\eta) \in C([0,T], H)$.

(Then note that $\eta(r) = \int_0^r (r-s)^{-\alpha} T_{r-s}[q(s)]\,dW_s$

$\in L^{2p}([0,T], H)$. $\mathbb{P}$-a.s. as estimated above. And

$F(\eta)$ is well-def $\Rightarrow$ $Z(Y)$ has conti. modifi.)

1) F is BLO: We can prove $\sup_{[0,T]} \|F(\eta)(t)\|_H^{2p} \le$

$C_{\alpha,T,p} \int_0^T \|\eta(t)\|_H^{2p}\,dt$ similar as above.

2) To prove $F(\eta)$ is conti. Note $C([0,T], H)$

$\subset L^{2p}([0,T], H)$ is dense. So $\exists (\eta_n)$ conti $\to$

$\eta \in L^{2p}$. $F(\eta_n) \to F(\eta)$ uniform at t by 1).

$\Rightarrow$ It's enough to prove $F(\eta)$ is conti.

for $\eta \in C([0,T], H)$.

$\|F(\eta)(t+h) - F(\eta)(t)\|_H \le$

$$\int_0^t \|[(t+h-r)^{\alpha-1} T_h - (t-r)^{\alpha-1}] T_{t-r}(\eta(r))\|_H\,dr +$$

$$\int_t^{t+h} (t+h-r)^{\alpha-1} \|T_{t+h-r}(\eta(r))\|_H\,dr \xrightarrow{h \to 0} 0$$

Since $\|T_{\square}(\eta(r))\| \le C$. uniformly $\mathbb{P}$-a.s.

left conti. can be proved similarly

(3) With additive noise:

In particular, if C is indept. of solution then the semilinear SPDE is called SPDE with additive noise $W_A(t) = \int_0^t T_{t-s} C \, dW_s$.

Let sol. $X_t = Y_t + W_A$. $\Rightarrow$ Y solves equation:

$dY_t = [AY_t + B(Y_t + W_A(t))]dt$. $Y_0 = \xi$.

Rmk: It can be used in num. approxi.