Variational Approach to SPDE

It means reducing "solve SPDE/PDE" to "find energy minimal point".

<u>e.g.</u> $J: \mathcal{X} \to \mathbb{R}'$. $f \mapsto \int_\Omega |\nabla f|^2 dx$. $\mathcal{X}$ is space of test func. Then: $f^* \in \arg\min J(f) \Rightarrow$ f^* solves $\Delta f = 0$.

<u>Pf:</u> $\forall \varepsilon > 0$. $\forall g \in \mathcal{X}$. $J(f^* + \varepsilon g) \geq J(f^*)$.

$\Rightarrow \int \langle \nabla f^*, \nabla g \rangle = \int \langle \Delta f^*, g \rangle = 0$.

(1) <u>For PDEs</u>:

First, we introduce Gelfand triple:

For H separa. $\mathbb{R}'$-Hilbert. $V \subseteq H$. subspace st. $V \overset{i}{\hookrightarrow} H$ is dense conti. embedding. Besides $\|h\|_H \leq \|h\|_V$ for $\forall h \in V$.

By Riesz isomorphism:

$H \cong H^* \overset{i^*}{\hookrightarrow} V^* \cong V$, $h \mapsto e_h \mapsto e_h|_V \mapsto \tilde{h}$.

the embedding i^* is also dense and conti.

Since $\|\ell_h\|_{V^*} = \sup_{\|v\|_V\le 1} |\ell_h(v)| = \sup_{\square} |\langle h, v\rangle_H|$

$\le \sup_{\square} \|h\|_H \|v\|_V \le \|h\|_H = \|\ell_h\|_{H^*}$

and Hahn-Banach Thm. $\Rightarrow$ it's dense.

e.g. $H^{1,2}(D) \hookrightarrow L^2(D) = L^2(D)^* \hookrightarrow H^{1,2}(D)^*$. D is open subset of $\mathbb{R}^d$.

$\Rightarrow$ We can identify $\Delta = \sum_i^d \partial_{x_i x_i}$ as BLO

by $u \in H^{1,2}(D) \mapsto (v \mapsto \langle \Delta u, v\rangle_{H^{1,2}}) \in H^{1,2}(D)^*$.

Where $\langle \Delta u, v\rangle = -\sum_i^d \int_D \partial x_i u \, \partial_{x_i} v \, dx$. $v \in H^{1,2}(D)$.

Rmk: Δ can be seen as extension on $C_c^2(D) \subset H^{1,2}(D)$ here.

Def: $A \in L(V, V^*)$ is coercive if $\exists \alpha, \lambda > 0$.

s.t. $-\langle Au, u\rangle \ge \alpha \|u\|_V^2 - \lambda \|u\|_H^2$. $\forall u \in V$.

Next, we consider IVP $(*)$: $A \in L(V, V^*)$.

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t) + f(t) & \text{on } \{t>0\}. \\ u(0) = u_0 \end{cases}$$

Lem*. If $u \in L^2([0,T], V)$. $t \mapsto u(t) \in AC([0,T], V^*)$ and weak deri. $du/dt \in L^2([0,T], V^*)$.

Thm: $u \in C([0,T], H)$ and dt-a.e.

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 = {}_{V^*}\langle \frac{du}{dt}(t), u(t)\rangle_V .$$

Pf: i) By Sobolev approxi. $\exists (u_m) \subset C^1([0,T], V)$. s.t. $\|u_m - u\|_{L^2(V)} + \|u_m' - u'\|_{L^2(V^*)} \to 0$

So $\lim_m \frac{d}{dt}\|u_m(t)\|_H^2 = \lim_m 2\langle u_m', u_m\rangle_V$

$= 2\langle u', u\rangle_V$. in L^1 by DCT

$\Rightarrow d\|u(t)\|_H^2/dt$ exists and $t \mapsto \|u(t)\|_H^2 \in AC[0,T]$ with the weak deri. $2\,{}_{V^*}\langle u', u\rangle_V$.

j) To prove: $u \in C([0,T], H)$.

Let $V \in L^2([0,T], H) \cap C^1([0,T], V)$ and $\langle V\rangle := \frac{1}{T}\int_0^T V(t)\,dt$. Then:

$$\|V(t)\|_H^2 \lesssim \|V - \langle V\rangle\|_H^2 + \|\langle V\rangle\|_H^2$$

$$\overset{\text{Hölder}}{\leq} \frac{2}{T}\int_0^T \|V(t) - V(r)\|_H^2\,dr + \frac{2}{T}\int_0^T \|V(r)\|_H^2\,dr$$

$$\|V(t) - V(r)\|_H^2 = 2\int_r^t \langle V'(s), V(s) - V(r)\rangle_V\,ds$$

$$\leq \int_0^T \|V'(s)\|_{V^*}^2\,ds + \int_0^T \|V(s)\|_V^2\,ds + T\|V(r)\|_V^2$$

So: $\sup_{[0,T]}\|V(t)\|_H^2 \lesssim \|V\|_{H^1([0,T],V)}^2$

Let $v = u_n - u_m$. $\Rightarrow u_n \xrightarrow{n} \tilde{u}$ in H.

st. $u = \tilde{u}$. a.e.

Rmk: Note V isn't closed. So $u \in C([0,T], H)$ only rather V.

Thm. For $A \in L(V, V^*)$ coercive. $u_0 \in H$. $f \in L^2([0,T], V^*)$. Then: IVP $(*)$ has unique sol. $u \in C([0,T], H) \cap L^2([0,T], V)$

Pf: i) Unique: let u, v sol's for $(*)$.

$$\Rightarrow d(u-v)/dt = A(u-v). \quad (u-v)(0) = 0.$$

$$\frac{1}{2} \frac{d}{dt} \|u-v\|_H^2 \overset{\text{Lem.}}{=} \langle A(u-v), u-v \rangle_V$$

$$\overset{\text{coer.}}{\leq} \lambda \|u-v\|_H^2$$

$$\text{So: } \frac{d}{dt} (e^{-2\lambda t} \|u-v\|_H^2) \leq 0$$

$$e^{-2\lambda t} \|u(t) - v(t)\|_H^2 \leq 0. \Rightarrow u = v.$$

2) Exist: Using Galerkin's method,

(e_n) is o.n.b. of H. $V_n \triangleq \mathrm{span}(e_k)_1^n$.

We solve $u_n \in C([0,T], V_n)$. st. $1 \leq k \leq n$

$$\frac{d}{dt} \langle u_n(t), e_k \rangle = \langle Au_n(t), e_k \rangle + \langle f(t), e_k \rangle$$

$$\langle u_n(0), e_k \rangle = \langle u_0, e_k \rangle.$$

Next we prove the estimate:

$$\sup_n \left(\sup_{[0,T]} \|u_n(t)\|_H^2 + \int_0^T \|u_n(t)\|_V^2 \, dt \right) < \infty$$

Since $\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_H^2 = \frac{1}{2}\frac{d}{dt}\sum_1^n \langle u_n, e_k\rangle^2$

$$= \sum_1^n {}_{V^*}\langle Au_n(t) + f(t), e_k\rangle \langle u_n(t), e_k\rangle_V$$

$$= {}_{V^*}\langle Au_n(t) + f(t), u_n(t)\rangle_V$$

$$\overset{\text{coer.}}{\leq} -\alpha\|u_n(t)\|_V^2 + \lambda\|u_n(t)\|_H^2 + \|f(t)\|_{V^*}\|u_n(t)\|_V$$

$$\overset{\text{AM-GM}}{\leq} -\frac{\alpha}{2}\|u_n(t)\|_V^2 + \lambda\|u_n(t)\|_H^2 + \frac{1}{2\alpha}\|f(t)\|_{V^*}^2.$$

Set $g(t) = \|u_n(t)\|_H^2 + \alpha\int_0^t \|u_n(s)\|_V^2 \, ds$

Integrate both sides of above on t.

$$\Rightarrow g(t) \leq \|u_0\|_H^2 + \frac{1}{\alpha}\int_0^t \|f(s)\|_{V^*}^2 ds + 2\lambda\int_0^t g(s) ds$$

Apply Gronwall's inequi. to conclude.

By boundedness, $\exists (u_{n_k}) \rightharpoonup u \in L^2([0,T], V)$.

$\Rightarrow Au_{n_k} \rightharpoonup Au$ in $L^2([0,T], V^*)$ since $A \in L(V, V^*)$. Also $u_n \xrightarrow{u} u$ in H.

So $u \in L^2(V) \cap C(H)$ solves $(*)$.

Generalization to Nonlinear op.:

Assume A is bdd. coercive but not linear.

Since the estimate still holds. So $\exists (u_{n_k}) \rightharpoonup u \in L^2([0,T], V)$. With coercive of A. (Au_n) is still L^2-bdd $\Rightarrow \exists (Au_{n_k}) \rightharpoonup g$ in $L^2(V^*)$.

But we can't conclude $g = Au$. here.

① Monotonicity method:

Assume $\theta \mapsto A((u+\theta w)(t))$ conti. & mono. condition:

$\exists \lambda \in \mathbb{R}^1$. $\langle Au - Av, u-v \rangle \leq \lambda \|u-v\|_H^2$

and bound condition: $\|Au\|_{V^*} \leq M(1+\|u\|_V)$

WLOG. assume $\lambda = 0$. (Or replace A by $A - \lambda I$.)

$\Rightarrow \int_0^T \langle Au_n - Av, u_n - v \rangle (t)\, dt \leq 0$. $\forall v \in L^2([0,T], V)$.

For $\int_0^T \langle Au_n, u_n \rangle (t)\, dt = \int_0^T \frac{d}{dt} \langle u_n, u_n \rangle - \langle u_n, f \rangle$

$= \frac{1}{2} (\|u_n(T)\|_H^2 - \sum_1^n \langle u_0, e_k \rangle^2) - \int_0^T \langle f, u_n \rangle dt$

Since $\exists u_{n_k} \rightharpoonup u$ in $L^2([0,T], V)$ with l.s.c. prop.

and $\int_0^T \langle Au_n, v \rangle + \langle Av, u_n \rangle$ also converges.

Besides, let $n \to \infty$: $\frac{d}{dt} \langle u(t), e_k \rangle = \langle g + f, e_k \rangle$. $\forall k$.

So: $\frac{d}{dt} u(t) = g(t) + f(t)$.

Apply Lemma* above, we have:

$\int_0^T \langle g(t), u(t)\rangle dt = \frac{1}{2}(\|u(T)\|_H^2 - \|u_0\|_H^2) - \int_0^T \langle f, u\rangle dt$

$\leq \overline{\lim} \int_0^T \langle Au_n, u_n\rangle(t)\, dt$

Similarly, for $\forall v \in L^2([0,T], V)$.

$\int_0^T \langle g(t) - Av(t), u(t) - v(t)\rangle$

$\leq \overline{\lim} \int_0^T \langle Au_n - Av, u_n - v\rangle(t)\, dt \leq 0.$

Let $v(t) = u(t) + \theta w(t)$. $\forall w \in L^2([0,T], V)$. $\theta > 0$.

$\Rightarrow \int_0^T \langle g(t) - A(u+\theta w), w\rangle(t)\, dt \leq 0.$

$\Rightarrow$ Set $\theta \to 0$. So: $g(t) = Au(t)$.

② Compactness method:

Assume $V \hookrightarrow H$ is cpt embedding and A is conti. from V to V^*.

<u>Lem</u>. $X := L^2([0,T], V) \cap H^{1,2}([0,T], V^*) \overset{cpt}{\hookrightarrow} L^2([0,T], H)$.

<u>Pf</u>: For $\varepsilon > 0$. Define $J_\varepsilon g(s) = \int_{s-\varepsilon}^{s+\varepsilon} g(t)\,dt / 2\varepsilon$

where we extend $g = g I_{[0,T]}$ on $\mathbb{R}^1$.

$\Rightarrow J_\varepsilon : L^2([0,T], V) \xrightarrow{BLO} C([0,T], V)$ from

$\|J_\varepsilon g\|_V \overset{\text{Hölder}}{\leq} \frac{1}{\sqrt{2\varepsilon}} \|g\|_{L^2([0,T], V)}$

i) Next, we show $J_\varepsilon : X \to C([0,T], H)$

is cpt by Ascoli Thm.

Note $\forall g \in X$. $\frac{d}{ds} J_\varepsilon g(s) = \frac{1}{2\varepsilon}(g(s+\varepsilon) - g(s-\varepsilon))$

$\|J_\varepsilon g(t_2) - J_\varepsilon g(t_1)\|_H = \|\int_{t_1}^{t_2} \frac{d}{ds} J_\varepsilon g(s) ds\|_H$

$\le \frac{1}{2\varepsilon} \int_{t_1}^{t_2} \|g(s+\varepsilon) - g(s-\varepsilon)\|_H ds$

$\overset{\text{Hölder}}{\le} \sqrt{t_2 - t_1} \|g\|_X / \varepsilon$.

If $A \subset X$ bdd. $\Rightarrow J_\varepsilon g$ is equicont. $\forall g \in A$.

So: $J_\varepsilon(A) \subset C([0,T], H)$ is relative cpt.

2) And $\|J_\varepsilon g(s) - g(s)\|_{V^*} = \|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} g(s+t) - g(s) dt\|_{V^*}$

$= \|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_0^t \frac{d}{du} g(s+u) du dt\|_{V^*}$

$\le \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \sqrt{|t|} (\int_0^t \|\frac{d}{du} g(s+u)\|_{V^*}^2 du)^{\frac{1}{2}} dt$

$\le 2\sqrt{\varepsilon} \|g\|_X / 3$.

$\Rightarrow \|J_\varepsilon g(s) - g(s)\|_{L^2([0,T],V^*)} \le 2\sqrt{\varepsilon T}\|g\|_X/3$.

3) For (g_n) bdd in X. $\Rightarrow \exists (g_{n_k}) \rightharpoonup g \in X$.

and $J_\varepsilon g_{n_k} \to J_\varepsilon g$ since J_ε is cpt. And

$\|u\|_H^2 = {}_{V^*}\langle u, u\rangle_V \le \delta \|u\|_V^2 + \frac{1}{\delta}\|u\|_{V^*}^2$. $\forall u \in V$.

So: $\|g_{n_k} - g\|_{L^2([0,T],H)}$

$\le \|g_{n_k} - J_\varepsilon g_{n_k}\|_\square + \|J_\varepsilon g_{n_k} - J_\varepsilon g\|_\square + \|J_\varepsilon g - g\|_\square$

$$\leq \delta \|g_{n_k} - J_\varepsilon g_{n_k}\|_{L^2(V)} + \delta^{-1}\|g_{n_k} - J_\varepsilon g_{n_k}\|_{L^2(V^*)}$$

$$+ \|J_\varepsilon(g_{n_k} - g)\|_{L^2(H)} + \delta\|g - J_\varepsilon g\|_{L^2(V)} +$$

$$\delta^{-1}\|g - J_\varepsilon g\|_{L^2(V^*)}$$

Next, we choose $\varepsilon > 0$. ($\sup_{n_k}\|J_\varepsilon g_{n_k} - g_{n_k}\| \xrightarrow{\varepsilon \to 0} 0$)

$\sup_k \delta^{-1}\|J_\varepsilon g_{n_k} - g_{n_k}\|_{L^2(V^*)} \vee \delta^{-1}\|J_\varepsilon g - g\|_{L^2(V^*)} < \delta$.

$\Rightarrow \overline{\lim_k}\|g_{n_k} - g\|_{L^2([0,T],H)} \leq 2\delta(\sup_k \|J_\varepsilon g_{n_k} - g_{n_k}\|_{L^2(V)} + 1)$

So: $g_{n_k} \to g$ in $L^2([0,T],H)$.

Therefore $\exists (u_{n_k}) \to u$ in $L^2([0,T],H)$. With conti. of A. $\Rightarrow Au_{n_k} \to Au$. $Au = g$.

(2) <u>For SPDE:</u>

① <u>Monotone - Coercive SPDEs:</u>

<u>Assumptions:</u>

Let $A: V \to V^*$ and $B: V \to L(U,H)$ satisfy

i) (Coercive) $\exists \alpha > 0$. λ, ν. s.t. $\forall u \in V$.

$$2\langle Au, u\rangle + \|B(u)\circ Q^{\frac{1}{2}}\|^2_{L_2(U,H)} \leq -\alpha\|u\|_V^2 + \lambda\|u\|_H^2 + \nu$$

ii) (Monotone) $\exists \lambda > 0$. s.t. $\forall u, v \in V$.

$$2\langle Au - Av, u - v\rangle + \|(B(u) - B(v))\circ Q^{\frac{1}{2}}\|^2_{L_2^0} \leq \lambda\|u - v\|_H^2$$

iii) (linear growth) $\exists M > 0$. s.t. $\forall u \in V$

$\|A(u)\|_{V^*} \le M(1 + \|u\|_V)$

iv) (Hemi-conti.) $\forall u, v, w$. $\lambda \mapsto \langle A(v + \lambda u), w \rangle$ conti.

Next, we consider to solve SPDE:

$du(t) = A(u(t))dt + B(u(t))dW_t$, $u(0) = u_0$.

Thm. Under assumpt. i) − iv). $\forall u_0 \in H$. There exists unique adapted process $u(t)$. s.t. $u \in L^2([0,T]; V) \cap C([0,T], H)$ and satisfy

$$\langle u(t), v \rangle = \langle u_0, v \rangle + \int_0^t \langle A(u(s)), v \rangle ds + \int_0^t \langle v, B(u(s))dW(s) \rangle.$$

Lem. (Itô's)

$u_0 \in H$. u, v are adapted $L^2([0,T], V)$. $L^2([0,T], V^*)$ paths. M is conti. H-valued local mart. s.t. $u(t) = u_0 + \int_0^t v(s) ds + M_t$.

Then: i) $u \in C([0,T], H)$ a.s.

ii) $\|u(t)\|_H^2 = \|u_0\|_H^2 + 2\int_0^t \langle v(s), u(s) \rangle ds + 2\int_0^t \langle u(s), dM_s \rangle + \langle M \rangle_t$ a.s.

Pf: $u_k(t) := \langle u(t), e_k \rangle = \langle u_0, e_k \rangle + \int_0^t v^k ds + M_t^k$.

where $V^k(s) = \langle V(s), e_k \rangle$. $M_t^k = \langle M_t, e_k \rangle$.

$\Rightarrow u_k(t) \in$ semimart. Apply Itô's:

$$du_k^2(t) = 2 \langle V(t), e_k \rangle u_k(t) dt + 2 u_k(t) dM_t^k + \langle M^k \rangle_t$$

So: $\|u(t)\|_H^2 = \sum u_k^2(t)$

$$\overset{DCT}{=} \sum \langle u_0, e_k \rangle^2 + 2 \int_0^t \sum V_k u_k ds$$

$$+ 2 \int_0^t \sum u_k(s) dM_s^k + \sum \langle M^k \rangle_t$$

$$= \|u_0\|_H^2 + \int_0^t 2 \langle V, u \rangle ds + 2 \langle u, dM \rangle + \langle M \rangle_t$$

For continuity of u:

Note $u \in C([0,T], V^*)$, since $u_0 \in H \subset V^*$.

and from ii): $t \mapsto \|u(t)\|_H^2$ is conti.

So we only need to prove $u(t)$ is weakly conti.: (H is uniform convex)

Since $\forall h \in H$. $\exists (h_n) \subset V$. st. $h_n \to h$ in H.

while $\langle u(t), h_n \rangle$ is conti. and $\langle u(t), h_n \rangle \xrightarrow{n} \langle u(t), h \rangle$ since $\sup_t \|u(t)\|_H < \infty$.

So: $\langle u(t), h \rangle$ is also t-conti. a.s.

<u>Cor.</u> $\varphi \in C^2(H; \mathbb{R}^1)$ in sense of Frechét

and $\varphi'(u) \in V$. $\forall u \in V$. st. $u \mapsto \varphi(u)$

is conti. w.r.t weak topo. Besides,

$\|\varphi'(u)\|_V \le M(1+\|u\|_V)$. Then:

$$\varphi(u(t)) = \varphi(u_0) + \int_0^t \langle v(s), \varphi'(u(s))\rangle ds + \int_0^t \langle \varphi'(u(s)), dM_s\rangle + \frac{1}{2}\int_0^t tr(\varphi''(u(s))Q(s))ds$$

where Q is Cov. matrix of M. i.e.

$$\langle\langle M, h\rangle, \langle M, g\rangle\rangle_t = \int_0^t \langle Q(s)h, g\rangle ds$$

Pf: $\varphi(u(t)) = \varphi(\sum^\infty u_k(t)e_k) = \tilde{\varphi}((u_k)_1^\infty)$

truncate φ and prove as above.

proof:

i) Uniqueness:

If u, v are two required sol.'s.

Set $\tau_n = \inf\{t \in [0,T] : \|u(t)\|_H^2 \vee \|v(t)\|_H^2 \vee (\int_0^t \|u(s)\|_V^2 \vee \|v(s)\|_V^2 ds) > n\} \uparrow T$.

Apply Lem' on $\|u-v\|_H^2$:

$$\|u-v\|_H^2 = \square \overset{\text{Assum ii)}}{\le} \int_0^t \langle u(s)-v(s), (Bu(s)-Bv(s))dW_s\rangle + \lambda\int_0^t \|u(s)-v(s)\|_H^2 ds$$

Take expectation. Since $\int_0^t \langle \square, \square dW\rangle$ is mart.

$$\Rightarrow \mathbb{E}(\|u_{t\wedge\tau_n} - v_{t\wedge\tau_n}\|_H^2) \le \lambda\int_0^t \mathbb{E}(\|u_\square - v_\square\|_H^2)ds$$

By Gronwall's: $\|u(t\wedge \tau_n) - v(t\wedge\tau_n)\|_H = 0$ a.s.

2) Existence:

Let $V_n = \text{span}(e_1, \dots, e_n)$ where (e_n) is o.n.b.

<u>Lem</u>[2]: $\forall n$, $\exists$ adapted $u_n \in C([0,T], V_n)$ s.t.

$$\langle u_n(t), e_k\rangle = \langle u_0, e_k\rangle + \int_0^t \langle A u_n(s), e_k\rangle ds + \int_0^t B_k(u_n(s))\, dW^n(s), \quad \forall 1\le k\le n.$$

where $B_k(u) h = \langle B(u) h, e_k\rangle$, $W^n(t) = \sum_1^n \langle W(t), f_k\rangle f_k$, (f_k) is o.n.b. of U.

<u>Pf</u>: Write the eq. into vector form

$$\tilde{u}_n(t) = \begin{pmatrix} \langle u_n(t), e_1\rangle \\ \vdots \\ \langle u_n(t), e_n\rangle \end{pmatrix} = \tilde{A}\, \tilde{u}_n(t)\, dt + \square \in \mathbb{R}^n$$

$\tilde{A}$, $\tilde{B}$ satisfy Lip. and linear growth

$\Rightarrow$ Apply basic Thm for SDE.

<u>Lem</u>[3]: $\sup_n \mathbb{E}\left(\sup_{[0,T]} \|u_n(t)\|_H^2 + \int_0^T \|u_n(s)\|_V^2 ds\right) < \infty$.

<u>Rmk</u>: Note that $\sup_n$ is outside. So: there's a little difference comparing to PDE case.

<u>Pf</u>: Apply Itô's on $\langle u_n(t), e_k\rangle_H^2$ and sum it up:

$$\|u_n(t)\|_H^2 = \sum_k^n \Big[\langle u_0, e_k\rangle^2 + 2\int_0^t \langle A u_n(s), e_k\rangle \langle u_n(s), e_k\rangle ds$$

$$+ 2\int_0^t \langle u_n(s), e_k\rangle B_k(u_n(s))\, dW_s^n$$

$$+ \sum_l^n \int_0^t \big(B_k(u_n(s))(\sqrt{\lambda_l} f_l)\big)^2 ds \Big]$$

$$\leq \|u_0\|_H^2 + \sum_1^n a_k + \sum_k \sum_l b_{k,l}$$

$$+ 2\int_0^t \langle u_n(s), B(u_n(s))\, dW_s^n\rangle.$$

$$\sum_k^n a_k := 2\int_0^t \langle A u_n(s), u_n(s)\rangle ds$$

$$\sum_k^n \sum_l^n b_{k,l} := \sum_k^n \sum_l^n \int_0^t \langle B(u_n(s)) \circ \sqrt{Q} f_l, e_k\rangle^2 ds$$

$$\leq \sum_l^n \int_0^t \langle B(u_n(s)) \circ \sqrt{Q} f_l\rangle^2 ds$$

$$\leq \int_0^t \| B(u_n(s)) \circ \sqrt{Q} \|_{L_2(U,H)}^2 ds$$

By assumpt i) and take $\mathbb{E}(\cdot)$:

$$\mathbb{E}(\|u_n(t)\|_H^2) \leq \|u_0\|_H^2 - \alpha\int_0^t \mathbb{E}(\|u_n(s)\|_V^2) ds$$

$$+ \lambda\int_0^t \mathbb{E}(\|u_n(s)\|_H^2) ds + \nu t$$

$$\leq \|u_0\|_H^2 + \nu T + \lambda\int_0^T \mathbb{E}(\|u_n(s)\|_H^2) ds$$

Apply Gronwall's inequ.

$$\mathbb{E}(\|u_n(t)\|_H^2) \leq (\|u_0\|_H^2 + \nu T)\, e^{\lambda T}.$$

And $\alpha\int_0^T \mathbb{E}(\|u_n(t)\|_V^2) dt \overset{\text{Assumpt i)}}{\leq}$

$$\|u_0\|_H^2 + \nu T + \lambda\int_0^T \mathbb{E}(\|u_n(t)\|_H^2) dt < \infty.$$

Similarly, we have:

$\sup_{[0,T]} \|u_n(t)\|_H^2 \le \|u_0\|^2 + VT + \sup_t \int_0^t \langle \square, \square\, dW_s^n \rangle$

Apply BDG inequi. on $\mathbb{E}(\sup_{[0,T]} \int_0^t \langle \square, \square\, dW_s^n \rangle)$

$\le C\,\mathbb{E}(\sup_t \|u_n(t)\|_H (\int_0^T \|B(u_n(s)) \circ Q^{\frac{1}{2}}\|_{L_2^0}^2 ds)^{\frac{1}{2}})$

$\overset{\text{Young}}{\le} \frac{1}{2}\mathbb{E}(\sup_{[0,T]} \|u_n(t)\|_H^2) + \frac{C}{2}\mathbb{E}(\int_0^T \|\cdots\|_{L_2^0} ds)$

$\overset{\text{Assum iii)}}{\le} \frac{1}{2}\mathbb{E}(\sup_{[0,T]} \|u_n(t)\|_H^2) + C\,\mathbb{E}(1 + \int_0^T \|u_n\|_V^2 ds)$

So: $\mathbb{E}(\sup_{[0,T]} \|u_n(t)\|_H^2) \le C$ (indept of n)

Return to proof of our Thm:

By Lem 3, (u_n) sol. seq in Lem 2 is bdd in

$L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega_T, V)$ $(\Omega_T = \Omega \times [0,T])$

With assumpt ii):

(Au_n) is bdd in $L^2(\Omega_T, V^*)$. $(B(u_n) \circ Q^{\frac{1}{2}})$ is

bdd in $L^2(\Omega_T; L_2(U, H))$.

So: $\exists (n_k)$ s.t. $u_{n_k} \to u$ in $L^2(\Omega_T, V)$. $Au_n \to g$

in $L^2(\Omega_T, V^*)$. $B(u_n) \circ Q^{\frac{1}{2}} \to \xi \circ Q^{\frac{1}{2}}$ in $L^2(\Omega_T; L_2(\square))$

Set $n \to \infty$. We have:

$\langle u(t), e_k \rangle = \langle u_0, e_k \rangle + \int_0^t \langle g(s), e_k \rangle ds + \int_0^t \langle e_k, \xi \circ Q^{\frac{1}{2}} dW_s \rangle$

Next, we prove $g = A(u)$ and $\xi \circ Q^{\frac{1}{2}} = B(u) \circ Q^{\frac{1}{2}}$.

We use monotonicity trick as before.

WLOG, let assumpt ii) holds with $\lambda = 0$ (or let $\tilde{A} = A - \lambda I$). Then:

$$Q_n \triangleq \int_0^T 2\langle A(u_n(s)) - A(V(s)), u_n(s) - V(s)\rangle$$

$$+ \|(B(u_n(s)) - B(V(s)))\circ Q^{\frac{1}{2}}\|^2_{L_2(H,H)}\,ds \leq 0.$$

Let $n \to \infty$. Since we have subseq convergence from above and

$$\mathbb{E}(\int_0^T 2\langle Au_n(s), u_n(s)\rangle + \|B(u_n(s))\circ Q^{\frac{1}{2}}\|^2_{L_2(H,H)}\,ds)$$

$$\geq \mathbb{E}(\|u_n(T)\|_H^2) - \|u_0\|_H^2$$

from Itô's on $\|u_n(T)\|^2$ (u_n is sol. for $\square$)

$\Rightarrow n \to \infty$ in eq^n: $\mathbb{E}(\int_0^T 2\langle \eta(s) - AV(s), u(s) - V(s)\rangle$

$$+ \|(\zeta - B(V(s)))\circ Q^{\frac{1}{2}}\|^2_{L_2(H,H)}\,ds) \leq \underline{\lim}_n Q_n \leq 0.$$

Let $V = u$. We have:

$\mathbb{E}(\|(\zeta - B(u(s)))\circ Q^{\frac{1}{2}}\|^2_{L_2(H,H)}\,ds) = 0 \Rightarrow \zeta\circ Q^{1/2} = B(u)\circ Q^{1/2}$

Then: $\mathbb{E}(\int_0^T 2\langle \eta(s) - AV(s), u(s) - V(s)\rangle) \leq 0$.

As before, with assumpt iv). let $V = u - \theta w$. and then $\theta \to 0$. $\Rightarrow \eta(s) = A(u(s))$

<u>e.g.</u> (Stoch. heat eq.)

Consider $\partial_t u = \frac{1}{2}\partial_{xx}u + \theta\partial_x u\,\partial_t W_t$. $u(0,x) = u_0(x)$

S_0 : $V = H^{1,2}(\mathbb{R}) \subset H = L^2(\mathbb{R}) \simeq L^2(\mathbb{R}) \subset (H^{1,2}(\mathbb{R}))^*$

$U = \mathbb{R}^1$, $Au = \frac{1}{2}\partial_{xx} u$. $\langle Au, u \rangle = -\frac{1}{2}\int (\partial_x u)^2 dx$

$B(u)h = \theta \partial_x u \cdot h$. $\|B(u)\circ Q^{\frac{1}{2}}\|^2_{L_2(\mathbb{R},H)} = \theta^2 \|\partial_x u\|^2_{L^2(\mathbb{R})}$

$\Rightarrow \langle Au, u \rangle + \|B(u)\circ Q^{\frac{1}{2}}\|^2_{L_2(\mathbb{R},H)} = (\theta^2 - 1)\|\partial_x u\|^2_{L^2(\mathbb{R})}$

Assumpt i) is satisfied $\Leftrightarrow$ $|\theta| < 1$.

For $\theta = \pm 1$. $u(t,x) = u_0(x \pm W_t)$ is an sol. if $u \in C^2(\mathbb{R})$, proved by Itô's.

Rmk: Here the sol. will inherit regularity of u_0. There's no regularization affect by noise in SDE.

For $\theta > 1$, it corresponds to a backward heat eq.: $\partial_t u = -k\,\partial_{xx} u$ (ill-posed)

② Compactness method:

Assumptions:

Let $A: V \to V^*$ and $B: V \to L(U,H)$ satisfy

i) (Coercive) $\exists \alpha > 0$. λ, ν. st. $\forall u \in V$.

$$2\langle Au, u \rangle + \|B(u)\circ Q^{\frac{1}{2}}\|^2_{L_2(U,H)} \le -\alpha\|u\|_V^2 + \lambda\|u\|_H^2 + \nu$$

ii) (linear growth) $\exists M > 0$. st. $\forall u \in V$

$$\|A(u)\|_{V^*} \le M(1 + \|u\|_V)$$

iii) (sublinear growth) $\exists M, \delta > 0$. s.t.

$\|B(u) \circ Q^{\frac{1}{2}}\|^2_{L_2(U,H)} \leq M(1 + \|u\|_V^{1-\delta})$

iv) $V \overset{cpt}{\hookrightarrow} H$

v) $u \mapsto A(u) : \sigma(V, V^*) \cap H \to \sigma(V^*, V)$ conti.

$u \mapsto B(u) : \sigma(V, V^*) \cap H \to L_2(U, H)$ conti.

Rmk: i) Recall Assumpt i) can imply:

$$\sup_n \mathbb{E}\left(\sup_{[0,T]} \|u_n(t)\|_H^2 + \int_0^T \|u_n(s)\|_V^2 ds\right) < \infty.$$

ii) $H^{1,2} \overset{cpt}{\hookrightarrow} L^2$ is an example for iv).

Def: p.m. $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is sol. for mart. problem associated with SPDE $(*)$:

$du(t) = A(u(t))dt + B(u(t))dW_t$. $u(0) = u_0$

if i) $\mathbb{P}(u(0) = u_0) = 1$.

ii) $M_t = u(t) - u(0) - \int_0^t A u(s) ds$ is conti. H-valued $\mathbb{P}$-mart. with

$\langle M \rangle_t = \int_0^t B(u(s)) \circ Q \circ B(u(s))^* ds$.

Rmk: Equi. def of ii) is: $\forall (e_k)$ o.n.b. of H. $\forall \varphi \in C_B^2(\mathbb{R}^1)$. we have $M_t^{i,\varphi}$ is conti. $\mathbb{R}^1$-valued mart. where

$$M_t^{i,\varphi} := \varphi(\langle u(t), e_i\rangle_H) - \varphi(\langle u_0, e_i\rangle) - \left(\int_0^t \varphi'(\langle u(s), e_i\rangle)\langle A(u(s)), e_i\rangle ds + \frac{1}{2}\int_0^t \varphi''(\langle u(s), e_i\rangle\langle (B\circ Q\circ B^*)(u(s))e_i, e_i\rangle ds\right).$$

Thm. Under assumpt i) – v) above. $\Rightarrow \exists$ sol. $\mathbb{P}$ to mart. problem for SPDE $(*)$.

Pf: As before, let (e_k) is o.n.b. of H. $V_n = \text{span}\{e_1 \dots e_n\}$. $\Pi_n : H \xrightarrow{\text{proj.}} V_n$. Note $V_n \cong \mathbb{R}^n$ so existence of sol. u_n for $\Pi_n(*) \Rightarrow$ existence of $\mathbb{P}_n$ solve mart. problem w.r.t. $\Pi_n(*)$. i.e. $\mathbb{P}_n = \mathcal{L}(u_n(t))$

$(0)_n$ $\text{supp}(\mathbf{P}_n) \subset C([0,T]; V_n)$
$(i)_n$ $\mathbf{P}_n(u(0) = \Pi_n u_0) = 1$
$(ii)_n$ $\forall \varphi \in C_b^2(\mathbf{R})$ and Φ_s continuous, bounded and $\mathcal{F}_s$-measurable

$$\mathbf{E}_n\left[(M_t^{i,\varphi} - M_s^{i,\varphi})\Phi_s\right] = 0.$$

Lem*: $(\mathbb{P}_n)$ is tight on $\Omega = C([0,T], \sigma(V^*, V)) \cap L^2([0,T]; V)$ w.r.t. topo's

$\tau_1 = \sigma(L^2([0,T], V), L^2([0,T], V))$

$\tau_2 =$ uniform topo. on $C([0,T], \sigma(V^*, \cdot))$

$\tau_3 =$ strong topo. on $L^2([0,T], H)$

Note Lem* implies $\exists (\mathbb{P}_{n_k})$. s.t.

$\mathbb{P}_{n_k} \xRightarrow{w} \mathbb{P}$ on $(\Omega, \tau_1 \vee \tau_2 \vee \tau_3)$

So: $\mathbb{P}_n \circ \pi_0^{-1} = \delta_{\pi_n u_0} \xrightarrow{w} \delta_{u_0}$. i.e. $\mathbb{P}$ will satisfy condition i) of Def.

As for ii): Note $w \in \Omega \mapsto (m_t^{i,\varphi}(w) - m_s^{i,\varphi}(w)) q_s(w)$ is conti. from assumpt v)

And from $\varphi \in C_B^2$ and assumpt ii), iii). intermediate value Thm. with Hölder's:

$|m_t^{i,\varphi} - m_s^{i,\varphi}| \le C_{t,s,\varphi,i,T}(1 + \int_0^T \|u(s)\|_V^2 ds)^{1-\delta}$.

So $\sup_n \mathbb{E}_n(|m_t^{i,\varphi} - m_s^{i,\varphi}|^{1/1-\delta}) < \infty$ by Rmk i).

$\Rightarrow \exists \, 1/1-\delta > p > 1$. st. $(m_t^{i,\varphi} - m_s^{i,\varphi}) q_s$ is u.i.

So: $\lim_n \mathbb{E}_n((m_t^{i,\varphi} - m_s^{i,\varphi}) q_s) = \mathbb{E}(\square) = 0$.

Pf for Lem 4:

1) For Z_1: $K_1 = \{u \in L^2([0,T], V) \mid \int_0^T \|u(t)\|_V^2 dt \le k\}$ is weakly cpt in $\sigma((L^2)^*, L^2) = \sigma(L^2, L^2)$

and $\sup_n \mathbb{P}_n(K_1^c) \le \frac{1}{k} \sup_n \mathbb{E}_n(\int_0^T \|u(t)\|_V^2 dt) \to 0$.

2) For Z_2: $K_2 = \{u \in C([0,T], V^*) \mid \sup_{[0,T]} \|u(t)\|_H \le k$, $\sup_{0 \le s \le t \le T} \|u(s) - u(t)\|_{V^*} / |t-s|^\beta \le k\}$ is cpt by Ascoli Thm. for some $\beta > 0$

$\mathbb{P}_n(K_2^c) \le \mathbb{P}_n(\sup \|u\|_H \ge k) + \mathbb{P}_n(\sup \frac{\|u(s) - u(t)\|}{\square} \ge k)$

Note for $\forall \|h\|_V \le 1$.

$$|\langle u(t), h\rangle - \langle u(s), h\rangle| \overset{\text{Assum ii)}}{\le}$$

$$M\left(|t-s| + |t-s|^{\frac{1}{2}} \int_s^t \|u(r)\|_V^2 \, dr\right) + \left|\int_s^t \langle h, \hat{B}(u(r))\, d\hat{W}(r)\rangle\right|$$

and $\mathbb{E}_n\left(\sup_{r\in[s,t]} \left|\int_s^r \langle h, \hat{B}(u(r))\, d\hat{W}(r)\rangle\right|^p\right)$

$$\overset{\text{BDG}}{\le} C_p \mathbb{E}_n\left(\left(\int_s^t \|B(u(r))\circ Q^{\frac{1}{2}}\|_{L_2^0}^2 \, dr\right)^{p/2}\right)$$

$$\overset{\text{Assum iii)}}{\le} C_p \mathbb{E}_n\left(\left(t-s + \int_s^t \|u(r)\|_V^{2(1-\delta)} \, dr\right)^{\frac{p}{2}}\right)$$

$$\overset{\text{Hölder}}{\le} C_{p,m}\left((t-s)^{p/2} + \left(\sup_n \mathbb{E}_n\left(\int_s^t \|u\|_V^2\right)\right)^{\square} \cdot (t-s)^{\frac{p}{2}\delta}\right)$$

$$\le C(t-s)^{\tilde{\delta}}.$$

$\Rightarrow$ We choose $\beta = \tilde{\delta} \wedge \frac{1}{2}$. $\int_0$: we have

$$\|u(t) - u(s)\|_{V^*} \le C(t-s)^{\beta}.$$

$$\mathbb{P}_n\left(\sup \frac{\|u(s) - u(t)\|}{\square} \ge k\right) \le C/k \to 0.$$

$$\Rightarrow \mathbb{P}_n(K_2^c) \le \mathbb{E}(\sup\|u\|^2)/k + C/k \to 0.$$

iii) For z_3:

<u>Lem.</u> (u_n) bdd in $L^2([0,T],V) \cap L^\infty([0,T],H)$ and equicontin$([0,T], V^*)$. st. $u_n(0) \to u_0$ in H. Then: $\exists$ strongly convergent subseq in $L^2([0,T],H)$

<u>Pf:</u> Recall $\|u\|_H \le \varepsilon\|u\|_V + \frac{C}{\varepsilon}\|u\|_{V^*}$. $\forall u\in V$

And by Ascoli Thm. $\exists$ subseq (u_{n_k}) $\to u \in C([0,T], V^*)$, so as in $L^2(V^*)$.

With Fatou's $\Rightarrow u \in L^2([0,T], V)$.

So: $\overline{\lim_{n\to\infty}} \int_0^T \|u_n(t) - u(t)\|_H^2 dt$

$\leq \overline{\lim_{n\to\infty}} \int_0^T \varepsilon \|\square\|_V^2 + \varepsilon^{-1} \|\square\|_{V^*}^2 dt$

$\leq \varepsilon T \sup_t \|u_n(t) - u(t)\|_V^2 + 0 \sim \varepsilon.$