Rough SPDEs

(1) <u>Motivation</u>:

Consider on $L^2([0,2\pi])$ with periodic bdy:

$$du = \partial_{xx} u\, dt + f(u)dt + g(u)\partial_x u\, dt + \sigma\, dW_t(x)$$

where $u \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^{n\times n}$. s.t. $f, g \in C^3$ and space-time white noise $W(s,x) = \sum_1^\infty \beta_k(s) e_k(x)$, $\beta_k \overset{i.i.d.}{\sim}$ SBM.

$\Rightarrow$ We want to give rigorous meaning for the SPDE above.

<u>Rmk</u>: Note "$g(u)\partial_x u$" is kind singular and can't be applied variational approach since $\nexists\ \gamma, m > 0$, s.t. $\int_0^{2\pi} g(u)\partial_x u \cdot u\, dx \le \gamma \|u\|^2_{H^{1,2}} + m\|u\|^2_{L^2}$.

$\Rightarrow$ It's not coercive.

Consider the mild approach $P_t = e^{t\partial_{xx}^2}$ and look for the mild sol.: $(n=1)$

$$u(t) = P_t u_0 + \int_0^t P_{t-s}(f(u_s) + g(u_s)\partial_x u_s)ds + \sigma\int_0^t P_{t-s}\, dW_s$$

Rmk: Recall ∂_{xx} has eigenvalues $-k^2$ with eigenvectors $e_k(x) = \pi^{-1/2}\cos(kx)$ & $f_k(x) = \pi^{-1/2}\sin(kx)$. (So they're also eigenvector for P_t with $e^{-k^2 t}$, $k \in \mathbb{Z}^{\geq 0}$)

For $W(t,x) = \sum \beta_k(s) e_k(x) + \tilde{\beta}_k(s) f_k(x)$ where $\beta_k, \tilde{\beta}_k \overset{i.i.d.}{\sim}$ SBM

$$\Rightarrow \int_0^t P_{t-s}\, dW_s = \beta_0(t)/\sqrt{2\pi}$$

$$+ \sum_{k=1}^{\infty} \int_0^t e^{-k^2(t-s)} (d\beta_k(s) e_k(x) + d\tilde{\beta}_k(s) f_k(x))$$

Recall $\|u\|_{H^{\alpha,2}} := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2\alpha}}\, dx\, dy$.

So: $\mathbb{E}(\|\int_0^t P_{t-s}\, dW_s\|^2_{H^{\alpha,2}})$

$$\overset{\text{expand}}{=} t + 2\sum_{k=1}^{\infty} k^{2\alpha} \int_0^t e^{-2k^2(t-s)}\, ds$$

$$= t + 2\sum_{k\geq 1} (1 - e^{-2k^2 t})/k^{2(1-\alpha)} < \infty$$

$$\Leftrightarrow \alpha < 1/2.$$

So no $\partial_x u$ can be expected to turn up in eq. of mild sol.

While we can consider the decomposition $u(t,x) = v(t,x) + \sigma W_{\partial_{xx}}(t)$ where $W_{\partial_{xx}}(t) = \int_0^t P_{t-s}\, dW_s$ and

$$V(t,x) = P_t u_0(x) + \int_0^t P_{t-s}\big(f(u(s)) + g(u(s))\partial_x V(s)\big)(x)\,ds + \sigma\int_0^t\int_0^{2\pi} P_{t-s}(x-y)\,g(u(s,y))\,\partial_x W_{\partial xx}(s,y)\,dy\,ds$$

Rmk: It's kinda variation of const. technique to consider $dW_{\partial xx}(s,x) = \frac{dW_{\partial xx}(s,y)}{dx}dx$

But note that there's some problem if interpret $\int_0^{2\pi} P_{t-s}(x-y)g(u(s,y))\,dW_{\partial xx}(s,y)$ as a sto-integral since the integrand is not adapted. So we need to introduce rough integral to define it.

(2) Stationary Solutions:

Consider $dX_t = (\partial_{xx}-1)X_t\,dt + \sigma\,dW_t$. (*)

The semigroup of $\partial_{xx}-1$ is $S_t := e^{-t}P_t$

Rmk: "-1" here is for the exponential decay factor e^{-t}.

$\Rightarrow X_t = \sigma\int_{-\infty}^t e^{-(t-s)}P_{t-s}\,dW_s$. $t\in\mathbb{R}'$ is the mild sol. where W_s is two side BM. (i.e. $W_t - W_s \sim N(0,|t-s|I)$. $\forall t,s\in\mathbb{R}'$)

X_t is also a stationary sol. to (*), i.e. its limit sol. X_∞ has stationary dist.

$\sim N(0, \sigma^2 \int_0^\infty e^{-2t} P_{2t}\, dt)$

Note that $\sigma^2 \int_0^\infty e^{-2t} P_{2t} f\, dt$

$= \sigma^2 \int_0^\infty e^{-2t} dt \cdot \langle f, e_0 \rangle e_0$

$\quad + \sum_{k \geq 1} \sigma^2 \int_0^\infty e^{-2(1+k^2)t} dt \, (\langle f, e_k \rangle e_k + \langle f, f_k \rangle f_k)$

$= \frac{\sigma^2}{2} \langle f, e_0 \rangle e_0 + \sum_{k \geq 1} \frac{\sigma^2}{2(1+k^2)} (\langle f, e_k \rangle e_k + \langle f, f_k \rangle f_k)$

$= \int f(y) K(\cdot - y) dy.$

Where $K(x-y) = \frac{\sigma^2}{2\pi} \sum_{k} \frac{\cos(kx)}{1+k^2} = \frac{\sigma^2}{2} \frac{\cosh(|x|-\pi)}{\sinh(\pi)}$

<u>Rmk:</u> We can see

$$\int_0^\infty e^{-2t} P_{2t} f\, dt = \frac{1}{2} (Id - \partial_{xx})^{-1} f$$

With $\mathbb{E}(|\langle X_t, f \rangle_{L^2} - \langle X_t, g \rangle_{L^2}|^2)$

$$= \int_0^{2\pi} (f-g)(x)\, Q(f-g)(x)\, dx$$

$$= \int_0^{2\pi} \int_0^{2\pi} (f-g)(x) (f-g)(y) K(x-y)\, dx\, dy.$$

We choose $f = \delta_x$ and $g = \delta_y$. Then:

$\mathbb{E}(|X_t(x) - X_t(y)|^2) \leq 2(K(0) - K(x-y)) \in Lip.$

Then, by Kolmogorov criteria, we have

$\forall$ fix t, $X_t(x)$ can be lifted to rough path $(X_t(x), \mathbb{X}_t(x,y))$

where $\mathbb{X}_t(x,y) \overset{a.s.}{=} \lim_{\varepsilon\to 0} \int_x^y (X_t^\varepsilon(v) - X_t^\varepsilon(x))\, \partial_x X_t^\varepsilon(v)\, dv$

$(X_t^\varepsilon(x))_x$ is Gaussian process, smooth on x.

and $\sup_x \mathbb{E}(|X_t^\varepsilon(x) - X_t(x)|^2) \to 0$.

And $\forall \alpha \in (0, \frac{1}{2})$, $\exists\, \theta, C_q > 0$, s.t.

$$\mathbb{E}(\|\delta X_{s,t}\|_\alpha^{2q} + \|\delta \mathbb{X}_{s,t}\|_{2\alpha}^{q}) \le C_q |t-s|^{\theta q}, \quad \forall q \ge 1.$$

(3) <u>Existence & Unique of Sol.:</u>

Let $\mathbf{X} = (X, \mathbb{X})$ is the rough path in (2). Let $v_t = u_t - X_t$, then we have:

$$v(t,x) = e^{-t} P_t(u_0 - X_0)(x)$$
$$+ \int_0^t e^{-(t-s)} (g(u(s,\cdot))\, \partial_x v(s,\cdot) + f(u(s,\cdot)) + u(s,\cdot))\, ds$$
$$+ \int_0^t \int_0^{2\pi} e^{-(t-s)} P_{t-s}(x-y)\, g(u(s,y))\, d\mathbf{X}_s(y)\, ds.$$

<u>Thm.</u> Let $\beta \in (\frac{1}{3}, \frac{1}{2})$, $u_0 \in C^\beta$. Then, a.s. every realization of X_t, $\exists\, T > 0$, s.t.

$du = \partial_{xx} u\, dt + f(u)\, dt + g(u)\, \partial_x u\, dt + \square$ in (1)

has a unique mild sol. taking the values in $C([0,T]; C^\beta)$. If additionally $f, g \in C_b^3$

then the sol. is global.

<u>Pf</u>: We fix $\alpha \in (\frac{1}{3}, \beta)$. Then we apply the Picard iteration on

$$V(t,x) = u(t,x) - X_t(x) - e^{-t} P_t(u_0 - X_0)(x)$$

$$= \int_0^t e^{-(t-s)} P_{t-s}\big(g(u(s,\cdot))(\partial_x V(s,\cdot) + \partial_x X(s,\cdot))$$

$$+ f(u(s,\cdot)) + X(s,\cdot)\big)(x)\,ds$$

$$+ \int_0^t \int_0^{2\pi} e^{-(t-s)} P_{t-s}(x-y)\, g(u(s,y))\, d\mathbb{X}_s(y)\, ds$$

$$\triangleq \mathcal{M}^{(1)}_{T,x} V(t,x) + \mathcal{M}^{(2)}_{T,x} V(t,x). \quad T \le 1.$$

We claim:

i) $\sup_{t\le T} \|\mathcal{M}^{(1)}_{T,x} V(t,x)\|_{C^1} \le C_K (1 + |||\mathbb{X}|||_\alpha) T^{\frac{\beta}{2}}$

$\sup_{t\le T} \|\mathcal{M}^{(1)}_{T,x} V - \mathcal{M}^{(1)} \bar{V}\|_{C^1}$

$\le C_K (1 + |||\mathbb{X}|||_\alpha) \sup_{t\le T} \|V - \bar{V}\|_{C^1}\, T^{\beta/2}$

ii) $\sup_{t\le T} \|\mathcal{M}^{(2)}_{T,x} V(t,x)\|_{C^1} \le C_K (1 + |||\mathbb{X}|||_\alpha)^3\, T^{\frac{\beta-\alpha}{2}}$

$\sup_{t\le T} \|\mathcal{M}^{(2)}_{T,x} V - \mathcal{M}^{(2)} \bar{V}\|_{C^1}$

$\le C_K (1 + |||\mathbb{X}|||_\alpha) \sup_{t\le T} \|V - \bar{V}\|_{C^1}\, T^{\frac{\beta-\alpha}{2}}$

$\Rightarrow$ We choose $T > 0$ small enough.

<u>Rmk</u>: We only need $f \in C^1$ and $g \in C^3$ for well-posedness of local sol.

Cor. (φ_ε) is molifier, $Q_\varepsilon f(x) = \varphi_\varepsilon * f(x)$

Then $du_\varepsilon = \partial_{xx} u_\varepsilon dt + f(u_\varepsilon) dt + g(u_\varepsilon)\partial_x u_\varepsilon dt + \sigma Q_\varepsilon dW_t$

has unique mild sol. u_ε on $[0, \tau_\varepsilon]$

s.t. $u_\varepsilon = V_\varepsilon + \sigma W^\varepsilon_{\partial_{xx}}$. $W^\varepsilon_{\partial_{xx}}(t) = \int_0^t P_{t-s} Q_\varepsilon dW_s$

and V_ε solves

$$V_\varepsilon(t,x) = P_t u_0(x) + \int_0^t P_{t-s}\big(f(u_\varepsilon(s,\cdot)) + g(u_\varepsilon(s,\cdot))\partial_x V_\varepsilon(s,\cdot)\big)(x) ds + \int_0^t \int_0^{2\pi} P_{t-s}(x-y) g(u_\varepsilon(s,y))\partial_x W^\varepsilon_{\partial_{xx}}(s,y) ds dy$$

Rmk: i) Here $\partial_x W^\varepsilon_{\partial_{xx}} \in L^2$ with

$$\mathbb{E}(\|\partial_x W^\varepsilon_{\partial_{xx}}(s)\|^2_{L^2}) \le 2\pi\varepsilon^{-3} \int \varphi'(x)^2 dx$$

So the last integral exists in classical sense

ii) $\exists\, \hat{\tau}_\varepsilon \le \tau_\varepsilon$, $\hat{\tau}_\varepsilon \to \tau$ a.s. s.t.

$$\lim_{\varepsilon\to 0} \sup_{t \le \hat{\tau}_\varepsilon} \|u_\varepsilon(t) - u(t)\|_{L^p} = 0.$$

which follows from the stab. result below by setting

$X^\varepsilon_t = \sigma \int_{-\infty}^t e^{-(t-s)} P_{t-s} Q_\varepsilon dW_s$ with

$\mathbb{X}^\varepsilon_t(x,y) = \int_x^y \delta X^\varepsilon_t(x,v) \partial_x X^\varepsilon_t(v) dv$

and notice $\mathbf{X}^\varepsilon_t \xrightarrow{u} \mathbf{X}_t$ in $\mathcal{C}^\alpha$

<u>Thm.</u> (Stability)

$u_0, \bar{u}_0 \in C^\beta$. $\mathbf{X}, \bar{\mathbf{X}} \in \mathcal{C}^\alpha$ with $\frac{1}{3} < \alpha < \beta < \frac{1}{2}$

Let $u, \bar{u}$ are corresph. local s.l. up to $\tau, \bar{\tau}$, then $\forall T < \tau \wedge \bar{\tau}$. $\forall \varepsilon > 0$. $\exists \delta > 0$. s.t.

$\sup_{t \le T} \|u_t - \bar{u}_t\|_{C^\beta} \le \varepsilon$.

for $\forall \bar{u}_0, \bar{\mathbf{X}}$. s.t. $\|u_0 - \bar{u}_0\|_{C^\beta} + \sup_{t \le T} |||\mathbf{X} - \bar{\mathbf{X}}|||_\beta \le \delta$.