On the Rough Heston and Quadratic Rough Heston Models

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Table

Introduction on Rough Heston Model

The rough Heston model, introduced in [ER19], has the form

$$dS_t = \sqrt{V_t} S_t \left(\rho dW_t + \sqrt{1-\rho^2} dB_t \right), \qquad S_0 = S_0, \tag{1}$$

$$V_t = V_0 + \int_0^t K(t-s)(\theta - \lambda V_s) ds + \int_0^t K(t-s) \nu \sqrt{V_s} dW_s, \tag{2}$$

where K is the fractional kernel

$$K(t) := \frac{t^{H-1/2}}{\Gamma(H+1/2)} = \int_0^\infty e^{-xt} \mu(dx), \tag{3}$$

$$\mu(dx) := \frac{x^{-H-1/2} dx}{\Gamma(H+1/2)\Gamma(1/2-H)},$$

for all $t > 0$, with Hurst parameter $H \in (-1/2, 1/2)$.

Weak Markovian Approximation

Approximate K by a discrete sum of exponential functions K^N given by

$$K^N(t) := \sum_{i=1}^{N} w_i e^{-x_i t}, \tag{4}$$

for some non-negative nodes $(x_i)_{i=1}^N$ and non-negative weights $(w_i)_{i=1}^N$. Using the approximation K^N of K, we can define the approximation (S^N, V^N) of (S, V) by

$$dS_t^N = \sqrt{V_t^N} S_t^N \left(\rho dW_t + \sqrt{1-\rho^2} dB_t \right), \qquad S_0 = S_0,$$

$$V_t^N = V_0 + \int_0^t K^N(t-s)(\theta - \lambda V_s^N) ds + \int_0^t K^N(t-s) \nu \sqrt{V_s^N} dW_s.$$

SDE for Weak Markovian Approximation

Besides, V^N can also be represented into solution to an N-dimensional stochastic differential equation by using [AK21, Proposition 2.1]:

$$dS_t^N = \sqrt{V_t^N} S_t^N \left(\rho dW_t + \sqrt{1-\rho^2} dB_t \right), \quad S_0 = S_0, \tag{5}$$

$$dV_t^{(i)} = -x_i (V_t^{(i)} - v_0^i) dt + (\theta - \lambda V_t^N) dt + \nu \sqrt{V_t^N} dW_t, \quad V_0^{(i)} = v_0^i, \tag{6}$$

for $i = 1, \dots, N$, satisfying $V_t^N = \boldsymbol{w} \cdot \boldsymbol{V}_t^N$, $\boldsymbol{w} \cdot \boldsymbol{v}_0 = V_0$ and $\boldsymbol{V}^N$ is an N-dimensional diffusion.

Remark

When $N = 1$, then it becomes classical Heston model.

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Error Bound for Markovian Approximation

Theorem ([BB23b, Corollary 2.12])

Under some assumptions on K. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be 8 times weakly differentiable and compactly supported. Then there is $C > 0$ such that

$$\left|\mathbb{E}h(S_T) - \mathbb{E}h(S_T^N)\right| \leq C \int_0^T \left|K(t) - K^N(t)\right| dt.$$

Theorem ([BB23b, Theorem 3.16])

For constants $\beta_0 = 0.92993273$, and $c_0 = 3.60585021$. Let K^N be a Gaussian approximation coming from a non-geometric Gaussian rule with parameters $c \geq c_0$, $\beta \geq \beta_0$, and $a > 0$, where $c > c_0$ or $\beta > \beta_0$. Then,

$$\begin{aligned}\int_0^T \left|K(t) - K^N(t)\right| dt &\lesssim \frac{c_H}{H + 1/2} \xi_n^{-1/2-H} \\ &= \Omega(1) \exp\left(-\frac{\eta_1(c, \beta)}{\beta}\sqrt{(H + 1/2)N}\right).\end{aligned}$$

Weak Scheme

Recall the SDEs for weak Markovian approximation of the volatility is

$$dV_t^{(i)} = -x_i(V_t^{(i)} - v_0^i)dt + (\theta - \lambda V_t^N)dt + \nu\sqrt{V_t^N}dW_t,$$
$$V_0^{(i)} = v_0^i, \qquad i = 1, \dots, N.$$

Write the SDEs into vector form

$$d\boldsymbol{V}_t^N = -\mathrm{diag}(\boldsymbol{x})\left(\boldsymbol{V}_t^N - \boldsymbol{v}_0\right)dt + \left(\theta - \lambda \boldsymbol{w}^\top \boldsymbol{V}_t^N\right)\mathbf{1}dt + \nu\sqrt{\boldsymbol{w}^\top \boldsymbol{V}_t^N}\mathbf{1}\, dW_t \tag{7}$$

Using the idea in [LM20] to split the SDE into two parts:

$$d\boldsymbol{Z}_t^N = -\mathrm{diag}(\boldsymbol{x})\left(\boldsymbol{Z}_t^N - \boldsymbol{v}_0\right)dt + \left(\theta - \lambda \boldsymbol{w}^\top \boldsymbol{Z}_t^N\right)\mathbf{1}\, dt,$$

$$d\boldsymbol{Y}_t^N = \nu\sqrt{\boldsymbol{w}^\top \boldsymbol{Y}_t^N}\mathbf{1}\, dW_t \tag{8}$$

Splitting method

1. For the deterministic part:

$$dZ_t^N = -\mathrm{diag}(\boldsymbol{x})\left(Z_t^N - \boldsymbol{v}_0\right)dt + \left(\theta - \lambda \boldsymbol{w}^\top Z_t^N\right)\mathbf{1}\,dt,$$

$$\widehat{D}(\boldsymbol{z},h) := D(\boldsymbol{z},h) = \boldsymbol{Z}_h = e^{Ah}\boldsymbol{z} + A^{-1}(e^{Ah} - \mathrm{Id})b,$$

2. For the random part: $d\boldsymbol{Y}_t^N = \nu\sqrt{\boldsymbol{w}^\top \boldsymbol{Y}_t^N}\mathbf{1}\,dW_t$

The strategy is to consider the inner product of the SDE (8) with $\boldsymbol{w}$ to get the SDE for $Y_t := \boldsymbol{w}^\top \boldsymbol{Y}_t^N$

$$dY_t = \nu\overline{w}\sqrt{Y_t}dW_t, \qquad Y_0 = \boldsymbol{w}\cdot\boldsymbol{y}.$$

$\rightsquigarrow$ As in [LM20], apply moment matching on Y_t to get a second order scheme $\widehat{Y}_t$.
To recover the approximation for $\boldsymbol{Y}_t^N$, we assert: $\widehat{Y}_h^i = y^i + Q, \qquad i = 1,\ldots,N.$
Then we solve $Q = \frac{\widehat{Y}_h - \boldsymbol{w}\cdot\boldsymbol{y}}{\overline{w}}$ and obtain a scheme for $\boldsymbol{Y}_t^N$, defined by

$$\widehat{S}(\boldsymbol{y},h) := \widehat{\boldsymbol{Y}}_h := \boldsymbol{y} + \frac{\widehat{Y}_h - \boldsymbol{w}\cdot\boldsymbol{y}}{\overline{w}},$$

where $\overline{w} := \sum w_i$. Finally, we combine the schemes and get

$$\widehat{A}(\boldsymbol{v},h) := \widehat{D}\left(\widehat{S}\left(\widehat{D}\left(\boldsymbol{v},\frac{h}{2}\right),h\right),\frac{h}{2}\right)$$

Weak Rate for the scheme

$\widehat{X}$ is a *potential weak ν-th-order scheme* for the operator L if for any $f \in C^\infty$,

$$R_{\nu+1}^{\widehat{p}(t)} f(x) = \mathbb{E}\left[f(\widehat{X}_t^x)\right] - \left[f(x) + \sum_{k=1}^{\nu} \frac{1}{k!} t^k L^k f(x)\right] = O(t^{\nu+1}).$$

It has uniformly bounded moments if

$$\exists n_0 \in \mathbb{N}^+, \forall q \in \mathbb{N}^+, \quad \sup_{n \geq n_0,\, 0 \leq i \leq n} \mathbb{E}\left[\|\widehat{X}_{t_i^n}^n\|^q\right] < \infty.$$

Theorem

$\widehat{A}$ is of strongly potential second order weak convergence.

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Proof

Def: $V^{N,M}_{t_{i+1}} := A(V^{N,M}_{t_i}, t_{i+1} - t_i)$ for $0 \leq t_0 \leq ... \leq t_M \leq T$.

i Well-definedness: $V^{N,M} \geq 0$? $\rightsquigarrow$ [AJBB24].

ii $(V^{N,M})_M$ has uniformly bounded moments
$\rightsquigarrow$ $\widehat{D}$ is linear & $\widehat{Y}$ is just three valued r.v. from moment matching.

iii To prove $\widehat{A}$ has potential second order rate:

Lemma

Let us consider $\widehat{p}^1_x, \ldots, \widehat{p}^m_x$ m potential second order discretization schemes on $\mathbb{D} \subset \mathbb{R}^n$ for the generators $L_1, \ldots, L_m$. Then, the transition probabilities

$$\widehat{p}^m(t/2) \circ \cdots \circ \widehat{p}^2(t/2) \circ \widehat{p}^1(t) \circ \widehat{p}^2(t/2) \circ \cdots \circ \widehat{p}^m(t/2) \tag{9}$$

are potential second order for the generator $\Sigma^m_{i=1} L$, where

$$\widehat{p}^2(t_2) \circ \widehat{p}^1_x(t_1)(dz) = \int_{\mathbb{D}} \widehat{p}^1_y(t_2)(dz)\, \widehat{p}^1_x(t_1)(dy).$$

Proof

So, we only need to check $\widehat{S}$ has potential second order rate, i.e.,

$$R_3^{\widehat{S}} f(\boldsymbol{y}) := \mathbb{E}(f(\widehat{\boldsymbol{Y}}_t^N)) - [f(\boldsymbol{y}) + tLf(\boldsymbol{y}) + \frac{t^2}{2} L^2 f(\boldsymbol{y})] = O(t^3),$$

where $Lf(\boldsymbol{x}) = \frac{\nu^2}{2} \sum_{i,j} (\boldsymbol{w}^\top \boldsymbol{x}) \partial_i \partial_j f(\boldsymbol{x})$ is generator for $\boldsymbol{Y}^N$.

Lemma (Generalization)

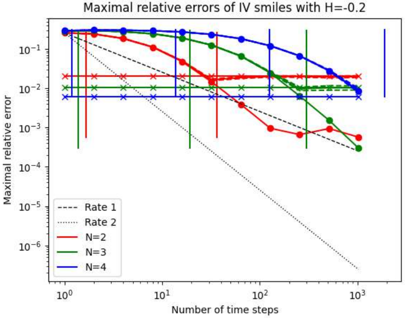
If $\sigma \in C^2(\mathbb{R}; \mathbb{R}^{\geq 0})$, $\widehat{\boldsymbol{Y}}$ is a weak scheme for the solution of a more general SDE

$$d\boldsymbol{Y}_t^N = \eta \sqrt{\sigma(\boldsymbol{w}^\top \boldsymbol{Y}_t^N)}\, \mathbf{1}\, dW_t, \tag{10}$$

obtained in a same way as above. And we assume $\widehat{Y} := \boldsymbol{w}^\top \cdot \widehat{\boldsymbol{Y}}$ is a potential second order approximation for the projected SDE $dY_t = \eta \sqrt{\sigma(Y_t)}\, \mathbf{1}\, dW_t$,where $Y_t = \boldsymbol{w}^\top \cdot \boldsymbol{Y}_t^N$. Then $\widehat{\boldsymbol{Y}}$ is a scheme of potential second order weak rate for $\boldsymbol{Y}^N$.

Numerics: Weak Simulation

Now we can verify the weak scheme for (S_t, V_t), proposed in [BB23a] is of weak rate two numerically. The following figure shows the maximal relative error of IV smiles for European call options with $H = -0.2$ under $N = 2, 3, 4$.



Introduction on Quadratic Rough Heston Model

The stock price S_t and volatility $V_t = p(Z_t)$ satisfy the dynamics

$$dS_t = S_t \sqrt{V_t} dW_t,$$

$$Z_t = \int_0^t K(t-s) b(Z_s) ds + \int_0^t K(t-s) \eta \sqrt{p(Z_t)} dW_s, \qquad (11)$$

where $b(x) = \theta - x$, $p(x) = a(x-b)^2 + c$, $a, b, c > 0$ and W is a Brownian motion, $H \in (0, \frac{1}{2})$.
For the log stock price $X_t = \log S_t$, the dynamics is

$$dX_t = -\frac{1}{2} p(Z_t) dt + \sqrt{p(Z_t)} dW_t, \qquad (12)$$

$$Z_t = \int_0^t K(t-s) b(Z_s) ds + \int_0^t K(t-s) \eta \sqrt{p(Z_t)} dW_s \qquad (13)$$

Error Bound for QRHM

Theorem (Error bound for the quadratic rough Heston model)

Replace K by approximation K^N in the model above to obtain X^N. If $\exists c > 0$, s.t., $K^N(t) \le cK(t)$ for any t, N. Then there exists a constant $C = C_H > 0$, for any payoff function $\phi \in C^3_{\mathrm{poly}}(\mathbb{R})$, we have

$$\begin{aligned} &|\mathbb{E}[\phi(X_T)] - \mathbb{E}[\phi(X_T^N)]| \\ &\le C \int_0^T \left\{ \|K - K^N\|_{L^1([0,t])} + \|K^2 - (K^N)^2\|_{L^1([0,t])} \right\} dt. \end{aligned} \tag{14}$$

Remark

Note that Z is a polynomial Volterra process (But (X, Z) isn't!), we have moment formula for it. Due to the structure of the moment formula, we can conjecture when $\phi(x) = x^k$ for any $k \in \mathbb{Z}^+$, it has such kind of estimation.

General Setting

We extend the PPDE method in [BJP25] which only works for the model below

$$X_t = x_0 + \int_0^t \psi(r, Z_r)\, dB_r + \zeta \int_0^t \psi^2(r, Z_r)\, dr, \tag{15}$$

$$Z_t = \int_0^t K(t-r)\, dB_r \tag{16}$$

to the model

$$X_t = x_0 + \int_0^t \psi(r, Z_r)\, dB_r + \zeta \int_0^t \psi^2(r, Z_r)\, dr, \tag{17}$$

$$Z_t = \int_0^t K(t-r)\, \big(b(Z_r)\, dr + \sigma(Z_r)\, dB_r\big), \tag{18}$$

where $b, \sigma \in C^1$ having bounded derivatives, $\psi \in C^{0,3}_{\text{poly}}([0, T] \times \mathbb{R})$ and $\partial_x \psi$ is bounded.

Main Result

Assumption

$K(t) \in C^1(0,T]$ is non-increasing and satisfies: there exists $H \in (\frac{1}{2}, 1)$,

$$|K(t-s)| \lesssim (t-s)^{H-1}, \quad |\partial_t K(t-s)| \lesssim (t-s)^{H-2},$$

for all $s < t$. Assume we also have another comparison kernel $\overline{K}(t-s)$, s.t., $\overline{K} \in C([0,T]; \mathbb{R}^{>0}) \cap L^2([0,T])$, and there exists $c_{K,\overline{K}} > 0$, for any $t \in (0,T]$,

$$\overline{K}(t) \leq c_{K,\overline{K}} K(t).$$

Theorem

Under all the assumptions above, replace the kernel K by $\overline{K}$ in our model to define the dynamics for $(\overline{X}, \overline{Z})$. Then, for any $\phi \in C^3_{\text{poly}}(\mathbb{R})$, there exists a constant $C \in \mathbb{R}_+$ (depending on T, H, ψ, ϕ, b, σ, $K|_{[0,T]}$, $\overline{K}|_{[0,T]}$)

$$|\mathbb{E}[\phi(X_T)] - \mathbb{E}[\phi(\overline{X}_T)]| \leq C \int_0^T \left\{ \|K - \overline{K}\|_{L^1([0,t])} + \|K^2 - \overline{K}^2\|_{L^1([0,t])} \right\} dt.$$

Representation in path-dependent function

Define the stochastic flow

$$\begin{aligned} X_T^{t,x,\omega} &= x + \zeta \int_t^T \psi^2(s, Z_s^{t,\omega})ds + \int_t^T \psi(s, Z_s^{t,\omega})dB_s, \\ Z_s^{t,\omega} &= \omega_s + \int_t^s K(s-r)\big(b(Z_r)\,dr + \sigma(Z_r)\,dB_r\big). \end{aligned} \tag{19}$$

Adjusted forward process

$$\Theta_s^t = \int_0^t K(s-r)\,\big(b(Z_r)\,dr + \sigma(Z_r)\,dB_r\big), \quad s \geq t. \tag{20}$$

Remark

Θ_s^t is a semimartingale for $s > t$. This process arises from the decomposition of Z as
$Z_s = \Theta_s^t + \int_t^s K(s,r)\,\big(b(Z_r)\,dr + \sigma(Z_r)\,dB_r\big) =: \Theta_s^t + I_s^t$.
In particular, $\Theta_t^t = Z_t$.

From [BJP25, Remark 2.15 & 2.17] $\rightsquigarrow$

$$\mathbb{E}[\phi(X_T)|\mathcal{F}_t] = u(t, X_t, \Theta_{[t,T]}^t).$$

Hadamard Derivative for u at (CTS) path variable

Lemma

The Hadamard derivatives of X at $\omega \in C^0([t, T])$ in the direction $\eta \in C^0([t, T])$:

(i) $\langle \partial_\omega X_T^{t,x,\omega}, \eta \rangle = \zeta \int_t^T \partial_x(\psi^2)(s, Z_s^{t,\omega})\eta_s ds + \int_t^T \partial_x \psi(s, Z_s^{t,\omega})\eta_s dB_s.$

(ii) $\langle \partial_{\omega\omega}^2 X_T^{t,x,\omega}, (\eta, \xi) \rangle = \zeta \int_t^T \partial_{xx}^2(\psi^2)(s, Z_s^{t,\omega})\eta_s \xi_s ds + \int_t^T \partial_{xx}^2 \psi(s, Z_s^{t,\omega})\eta_s \xi_s dB_s.$

Proposition

We have $u \in C_+^{2,2}$. And for $\xi, \eta \in C^0([t, T])$, we have:

$$\partial_x u(t,x,\omega) = \mathbb{E}[\phi'(X_T^{t,x,\omega})] \tag{21}$$

$$\partial_{xx}^2 u(t,x,\omega) = \mathbb{E}[\phi''(X_T^{t,x,\omega})] \tag{22}$$

$$\langle \partial_\omega u(t,x,\omega), \eta \rangle = \mathbb{E}[\phi'(X_T^{t,x,\omega})\langle \partial_\omega X_T^{t,x,\omega}, \eta \rangle] \tag{23}$$

$$\langle \partial_\omega \partial_x u(t,x,\omega), \eta \rangle = \mathbb{E}[\phi''(X_T^{t,x,\omega})\langle \partial_\omega X_T^{t,x,\omega}, \eta \rangle] \tag{24}$$

$$\langle \partial_{\omega\omega}^2 u(t,x,\omega), (\eta, \xi) \rangle = \mathbb{E}[\phi''(X_T^{t,x,\omega})\langle \partial_\omega X_T^{t,x,\omega}, \eta \rangle \langle \partial_\omega X_T^{t,x,\omega}, \xi \rangle] \tag{25}$$

$$+ \mathbb{E}[\phi'(X_T^{t,x,\omega})\langle \partial_{\omega\omega}^2 X_T^{t,x,\omega}, (\eta, \xi) \rangle]. \tag{26}$$

When the direction is singular

If the directions $Q_{(1)}, Q_{(2)} \in L^2$ might be singular at $t = 0$, how to develop the formulas for $\langle \partial_\omega u(t,x,\omega), Q_{(i)} \rangle$ and $\langle \partial^2_{\omega\omega} u(t,\omega), (Q_{(1)}, Q_{(2)}) \rangle$?

$\rightsquigarrow$ Following [BMM25], first denote $Q^t(r) := Q(r-t)$ for $u \in [t, T]$, so that $Q^t_{(i)} \in L^2([t,T], \mathbb{R}_+)$ and we define the truncated directions

$$Q^{t,\delta}(s) = Q^t(s \vee (t+\delta)) = \begin{cases} Q(s-t) & \text{if } s \geq t+\delta, \\ Q(\delta) & \text{if } s \in [0, t+\delta]. \end{cases}$$

Define the derivatives of u (if the limits exist)

$$\begin{aligned} \langle \partial_\omega u(t,x,\omega), Q^t_{(i)} \rangle &= \lim_{\delta \to 0} \langle \partial_\omega u(t,x,\omega), Q^{t,\delta}_{(i)} \rangle, \\ \langle \partial^2_{\omega\omega} u(t,\omega), (Q^t_{(1)}, Q^t_{(2)}) \rangle &= \lim_{\delta \to 0} \langle \partial^2_{\omega\omega} u(t,\omega), (Q^{t,\delta}_{(1)}, Q^{t,\delta}_{(2)}) \rangle. \end{aligned} \tag{27}$$

$\rightsquigarrow$ We also need the Malliavin Calculus to compute the (second-order) path derivatives.

When the direction is singular

Lemma

We have

$$\langle \partial_\omega X_T^{t,x,\omega}, Q_{(i)}^t \rangle = \zeta \int_t^T \partial_x(\psi^2)(s, Z_s^{t,\omega}) Q_{(i)}^t(s) ds + \int_t^T \partial_x \psi(s, Z_s^{t,\omega}) Q_{(i)}^t(s) dB_s$$

and

$$\langle \partial_\omega u(t,x,\omega), Q_{(i)}^t \rangle = \mathbb{E}[\phi'(X_T^{t,x,\omega}) \langle \partial_\omega X_T^{t,x,\omega}, Q_{(i)}^t \rangle], \tag{28}$$

$$\langle \partial_\omega \partial_x u(t,x,\omega), Q_{(i)}^t \rangle = \mathbb{E}[\phi''(X_T^{t,x,\omega}) \langle \partial_\omega X_T^{t,x,\omega}, Q_{(i)}^t \rangle], \tag{29}$$

$$\begin{aligned} \langle \partial^2_{\omega\omega} u(t,x,\omega), (Q_{(1)}^t, Q_{(2)}^t) \rangle = & \mathbb{E}[\phi''(X_T^{t,x,\omega}) \langle \partial_\omega X_T^{t,x,\omega}, Q_{(1)}^t \rangle \langle \partial_\omega X_T^{t,x,\omega}, Q_{(2)}^t \rangle \\ & + \phi'(X_T^{t,x,\omega}) \int_t^T \partial^2_{xx}(\psi^2)(s, Z_s^{t,\omega}) (Q_{(1)} Q_{(2)})^t(s) ds \\ & + \phi''(X_T^{t,x,\omega}) \int_t^T \mathbf{D}_s X_T^{t,x,\omega} \ \partial^2_{xx}\psi(s, Z_s^{t,\omega}) \qquad (30) \\ & \times (Q_{(1)} Q_{(2)})^t(s) ds]. \end{aligned} \tag{31}$$

Proof

Lemma (Moment Estimations)

For any $p \geq 1$ *and* $(x, \omega) \in \mathbb{R} \times C^0([0,T], \mathbb{R})$*. There exists positive constants* $c_1, c_2, \overline{c}_1$ *and* $\overline{c}_2$ *depending on* $K|_{[0,T]}$, $\overline{K}|_{[0,T]}$, b, σ, ψ *and* T, *s.t.*

(i) *We have*

$$\sup_{t\in[0,T]} \mathbb{E}[|X_T^{t,x,\omega}|^p] \leq c_1\big(1 + |x|^p + \|\omega\|^p_{[0,T]}\big),$$

$$\sup_{t\in[0,T]} \sup_{s\in[t,T]} \mathbb{E}[|Z_s^{t,\omega}|^p] \leq c_2\big(1 + \|\omega\|^p_{[0,T]}\big),$$

$$\sup_{t\in[0,T]} \mathbb{E}[|X_T^{t,\overline{X}_t,\overline{\Theta}^t}|^p] \leq \overline{c}_1\big(|x|^p + 1\big),$$

$$\sup_{t\in[0,T]} \sup_{s\in[t,T]} \mathbb{E}[|Z_s^{t,\overline{\Theta}^t}|^p] \leq \overline{c}_2.$$

(ii) *We have*

$$\mathbb{E}[|\sup_{0\leq t\leq T} Z_t|^p] \vee \mathbb{E}[|\sup_{0\leq t\leq T} X_t|^p] < \infty.$$

Proof

And where we use the Malliavin calculus in the proof?
$\rightsquigarrow$The trickiest part is to obtain the formula for the second-order path derivative.
And the most important step in our proof is to show

$$\mathbb{E}_{t,x,\omega}\left[\phi'(X_T)\int_t^T \partial^2_{xx}\psi(s,Z_s)(Q^{\delta}_{(1)}Q^{\delta}_{(2)})(t-s)\,dB_s\right]$$
$$\to \mathbb{E}_{t,x,\omega}\left[\phi'(X_T)\int_t^T \partial^2_{xx}\psi(s,Z_s)(Q_{(1)}Q_{(2)})(t-s)\,dB_s\right].$$

NOTICE: $\int_t^T \partial^2_{xx}\psi(s,Z_s)(Q_{(1)}Q_{(2)})(t-s)\,dB_s$ may not be well-defined! Since we only assume $Q_{(i)} \in L^2$.

Using Malliavin integration by part:

$$\mathbb{E}\left[F\int_t^T h_s\,dB_s\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T h_s\mathbf{D}_s F ds\middle|\mathcal{F}_t\right], \tag{32}$$

for $F \in \mathbb{D}^{1,2}$ and $h \in L^2(\Omega\times[0,T],\mathbb{R})$.

Proof

$\rightsquigarrow$ Rewrite and prove

$$\mathbb{E}_{t,x,\omega}\left[\phi'(X_T)\int_t^T \partial_{xx}^2\psi(s,Z_s)(Q_{(1)}^{\delta}Q_{(2)}^{\delta})(t-s)\,dB_s\right] \tag{33}$$

$$= \mathbb{E}_{t,x,\omega}\left[\int_t^T \mathbf{D}_s\phi'(X_T)\partial_{xx}^2\psi(s,Z_s)(Q_{(1)}^{\delta}Q_{(2)}^{\delta})(t-s)ds\right] \tag{34}$$

$$\to \mathbb{E}_{t,x,\omega}\left[\int_t^T \mathbf{D}_s\phi'(X_T)\partial_{xx}^2\psi(s,Z_s)(Q_{(1)}Q_{(2)})(t-s)ds\right](\delta\to 0) \tag{35}$$

Using the chain rule, to compute $\mathbf{D}_s\phi'(X_T)(=\phi''(X_T)\mathbf{D}_sX_T)$

$\rightsquigarrow$ Compute $\mathbf{D}_sX_T \rightsquigarrow$ Compute $\mathbf{D}_sZ_T$

$\rightsquigarrow$ Difficulty: In [BJP25], $Z_t=\int_0^t K(t,r)\,dB_r \Rightarrow \mathbf{D}_sZ_t = \mathbf{1}_{s\le t}K(t,s)$.

Now $Z_t=\int_0^t K(t,r)\left(b(Z_r)\,dr+\sigma(Z_r)\,dB_r\right)$ has more complex structure here and it's impossible to compute its Malliavin derivative explicitly.

$\Rightarrow$ $\mathbf{D}_sZ_t$ may satisfy some equation and we can do some estimation?

Malliavin Calculus for Stochastic Volterra Equation

Lemma

The SVE solution $Z_t \in \mathbb{D}^{1,2}$ *for* $0 \leq t \leq T$ *and its Malliavin derivative* $\mathbf{D}_r Z_t$ *will satisfy*

$$\mathbf{D}_r Z_t = K(t,r)\sigma(Z_r) + \int_r^t K(t,s)\big(\partial_x b(Z_s)\mathbf{D}_r Z_s ds + \partial_x \sigma(Z_s)\mathbf{D}_r Z_s dB_s\big), \quad (36)$$

for $r \leq t$, *a.s., and* $\mathbf{D}_r Z_t = 0$ *for* $T \geq r > t$. *Besides we have the following estimate*

$$\sup_{0\leq r\leq T} \int_r^T \mathbb{E}(|\mathbf{D}_r Z_t|^2)dt < \infty. \quad (37)$$

Remark

The same result holds for the stochastic flow

$$Z_t^{t_0,w} = w + \int_{t_0}^t K(t,r)\left(b(Z_r)\,dr + \sigma(Z_r)\,dB_r\right). \quad (38)$$

Regularity property for $u(t, x, \omega)$

Corollary

$$\sup_{0\le r\le T}\int_r^T \mathbb{E}[|\mathbf{D}_r Z_t|^{(2+\epsilon_H)}]dt \le C_{K,T,H,b,\sigma} < \infty.$$

Corollary

We thereby have: for any $p \in [1, 2)$,

$$\sup_{0\le t\le T}\sup_{t\le s\le T}\|\mathbf{D}_s X_T^{t,x,\omega}\|_{L^p(\Omega)} \le C_{H,p,T,\psi,\|\omega\|_{[0,T]}} < \infty. \tag{39}$$

Along with the corollary, we can prove the result below by following the proof of [BJP25, Proposition 2.24].

Proposition

Under the assumptions above. We have $u \in C^{0,2,2}_{+,\frac{1}{2}}(\Lambda)$.

Path dependent Feynman-Kac formula and Functional Itô formula

Apply [BJP25, Proposition 2.13, 2.14], we obtain

1. Path dependent Feynman-Kac formula

$$\begin{aligned}\partial_t u(t,x,\omega) &+ \zeta\psi^2(t,\omega_t)\partial_x u(t,x,\omega) + b(\omega_t)\langle\partial_\omega u(t,x,\omega), K(\cdot - t)\rangle \\ &+ \psi(t,\omega_t)\sigma(\omega_t)\langle\partial_\omega(\partial_x u)(t,x,\omega), K(\cdot - t)\rangle + \frac{1}{2}\psi^2(t,\omega_t)\partial^2_{xx}u(t,x,\omega) \\ &+ \frac{1}{2}\sigma^2(\omega_t)\langle\partial^2_{\omega\omega}u(t,x,\omega), (K(\cdot - t), K(\cdot - t))\rangle = 0, \qquad u(T,x,\omega) = \phi(x).\end{aligned}$$

2.Functional Itô formula

$$\begin{aligned}u(t,\overline{X}_t,\overline{\Theta}^t) = u(0,X_0,0) &+ \int_0^t \partial_t u(s,\overline{X}_s,\overline{\Theta}^s)ds + \int_0^t \zeta\psi^2(s,\overline{Z}_s)\partial_x u(s,\overline{X}_s,\overline{\Theta}^s)ds \\ &+ \int_0^t b(\overline{Z}_s)\langle\partial_\omega u(s,\overline{X}_s,\overline{\Theta}^s), \overline{K}(\cdot - s)\rangle ds + \int_0^t \psi(s,\overline{Z}_s)\sigma(\overline{Z}_s)\langle\partial_\omega(\partial_x u)(s,\overline{X}_s,\overline{\Theta}^s), \overline{K}(\cdot - s)\rangle ds \\ &+ \frac{1}{2}\int_0^t \psi^2(s,\overline{Z}_s)\partial^2_{xx}u(s,\overline{X}_s,\overline{\Theta}^s)ds + \frac{1}{2}\int_0^t \sigma^2(\overline{Z}_s)\langle\partial^2_{\omega\omega}u(s,\overline{X}_s,\overline{\Theta}^s), (\overline{K}(\cdot - s), \overline{K}(\cdot - s))\rangle ds \\ &+ \int_0^t \psi(s,\overline{Z}_s)\partial_x u(s,\overline{X}_s,\overline{\Theta}^s)dB_s + \int_0^t \sigma(\overline{Z}_s)\langle\partial_\omega u(s,\overline{X}_s,\overline{\Theta}^s), \overline{K}(\cdot - s)\rangle dB_s.\end{aligned}$$

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Estimation

Following [BMM25], we can use the strategy to estimate the difference below

$$\begin{aligned}\mathcal{E}_T :&= \mathbb{E}[\phi(\overline{X}_T)] - \mathbb{E}[\phi(X_T)] \qquad (40)\\ &= \mathbb{E}[u(0,x,0) - u(T,\overline{X}_T,\overline{\Theta}^T)] \qquad (41)\end{aligned}$$

$\rightsquigarrow$ Apply the functional Itô formula and then use the Path dependent Feynman-Kac formula to replace $\partial_t u$ in the equation. Since all the Itô integrals are martingale (L^2-bounded), it can be reduced to the following equation

$$\begin{aligned}\mathcal{E}_T =&\mathbb{E}\Big(\int_0^T b(\overline{Z}_s)\langle\partial_\omega u(s,\overline{X}_s,\overline{\Theta}^s),\Delta K^s\rangle ds\\ &+\int_0^T \psi(s,\overline{Z}_s)\sigma(\overline{Z}_s)\langle\partial_\omega(\partial_x u)(s,\overline{X}_s,\overline{\Theta}^s),\Delta K^s\rangle ds\\ &+\frac{1}{2}\int_0^T \sigma^2(\overline{Z}_s)\langle\partial^2_{\omega\omega}u(s,\overline{X}_s,\overline{\Theta}^s),(\Delta K^s,\Sigma K^s)\rangle ds\Big),\end{aligned}$$

where $\Delta K^s := \overline{K}^s - K^s$ and $\Sigma K^s := \overline{K}^s + K^s$.

Estimation: Malliavin Calculus still matters

Replace the path derivatives inside the equation by using the formulas in the proposition above, we can now use Hölder's inequality, BDG inequality to do the estimations by following the idea in [BMM25]...

Remark

The constant C_H depends on H because it contains the term $\|K\|_{L^{2+\epsilon_H}}$.

The tricky part is to cope with the terms involving

$$\int_t^T \partial_x \psi(s, Z_s^{t,\overline{\Theta}^t}) \Delta K(s-t) dB_s.$$

Commonly, we use the BDG inequality to estimate it. But here the Itô isometry will lead to a time integral of $(\Delta K)^2$, which gives us an undesirable L^2 bound.
To overcome it, we instead resort to Malliavin calculus and use the conditional integration by part formula.

Estimation: An example

The trickiest term is

$$\int_0^T \mathbb{E}\left[\sigma^2(\overline{Z}_t)\mathbb{E}\left[\phi''(X_T^{t,\overline{X}_t,\overline{\Theta}^t})\left(\int_t^T \partial_x\psi(s, Z_s^{t,\overline{\Theta}^t})\Sigma K(s-t)dB_s\right)\right.\right.$$
$$\left.\left.\times\left(\int_t^T \partial_x\psi(s, Z_s^{t,\overline{\Theta}^t})\Delta K(s-t)dB_s\right) \;\Big|\; \mathcal{F}_t\right]\right]dt. \tag{42}$$

Using the conditional integration by part, we can rewrite it as

$$\int_0^T \int_t^T \mathbb{E}\left[\sigma^2(\overline{Z}_t)\mathbb{E}\left[\mathbf{D}_s\left\{\phi''(X_T^{t,\overline{X}_t,\overline{\Theta}^t})\int_t^T \partial_x\psi(r, Z_r^{t,\overline{\Theta}^t})\Sigma K(r-t)dB_r\right\}\right.\right.$$
$$\left.\left.\times\,\partial_x\psi(s, Z_s^{t,\overline{\Theta}^t}) \;\Big|\; \mathcal{F}_t\right]\right]\Delta K(s-t)dsdt. \tag{43}$$

$\rightsquigarrow$ Compute $\mathbf{D}_s\left\{\phi''(X_T^{t,\overline{X}_t,\overline{\Theta}^t})\int_t^T \partial_x\psi(r, Z_r^{t,\overline{\Theta}^t})\Sigma K(r-t)dB_r\right\}$ by chain rule.

$\rightsquigarrow$ Using the moment estimations for $X_T^{t,\overline{X}_t,\overline{\Theta}^t}$ and $Z_r^{t,\overline{\Theta}^t}$ to estimate.

Weak Simulation

As in the RHM, we're going to find an efficient weak scheme for the weak Markovian approximation

$$
\begin{aligned}
d\boldsymbol{Z}_t^N = & -\operatorname{diag}(\boldsymbol{x})\left(\boldsymbol{Z}_t^N - \boldsymbol{z}_0\right)dt + \left(\theta - \boldsymbol{w}^\top \boldsymbol{Z}_t^N\right)\mathbf{1}\,dt \\
& + \left(\eta\sqrt{a(\boldsymbol{w}^\top \boldsymbol{Z}_t^N - b)^2 + c}\right)\mathbf{1}\,dW_t.
\end{aligned}
$$

$\rightsquigarrow$ Use the splitting method to separate the deterministic part and stochastic part.
We only need to care on the stochastic part

$$
d\boldsymbol{Y}_t^N = \left(\eta\sqrt{a(\boldsymbol{w}^\top \boldsymbol{Y}_t^N - b)^2 + c}\right)\mathbf{1}\,dW_t. \tag{44}
$$

Problem: We can't reduce it to the CIR process by taking the inner product with w and use the same way of moment matching from [LM20] as in the rough Heston model.

Weak Talyor scheme

Rather, perform moment matching on the increments of BM. Define

$$\xi = \begin{cases} +\sqrt{3}, & p = 1/6, \\ 0, & p = 2/3, \\ -\sqrt{3}, & p = 1/6, \end{cases}$$

So ξ is a second order weak scheme for ΔB_1.
$\rightsquigarrow$ Use the second-order weak Taylor scheme provided in [KP92], we now have a scheme with weak rate 2: (Note $\sqrt{p(x)}$ is regular enough)

$$\begin{aligned} \widehat{Y}^i_{t_{n+1}} =& \widehat{Y}^i_{t_n} + \eta \sqrt{p(\widehat{\boldsymbol{Y}}^N_{t_n})}\,\sqrt{\Delta t}\,\xi + \frac{a\overline{w}\eta^2}{2}(w^\top \cdot \widehat{\boldsymbol{Y}}^N_{t_n} - b)[\Delta t\,\xi^2 - \Delta t] \\ &+ \eta^3\Big(\frac{a\overline{w}^2}{4}\sqrt{p(\widehat{\boldsymbol{Y}}^N_{t_n})} - \frac{(a\overline{w})^2}{4}\frac{(w^\top \cdot \widehat{\boldsymbol{Y}}^N_{t_n} - b)^2}{\sqrt{p(\widehat{\boldsymbol{Y}}^N_{t_n})}}\Big)(\sqrt{\Delta t})^3\,\xi, \end{aligned} \tag{45}$$

for $i = 1, 2, ..., N$. We denote this scheme by $\widehat{S}'(\boldsymbol{y}, \Delta t)$.

Weak Scheme for (S, V)

$\rightsquigarrow$ Define the scheme for Z^N by using the Strang splitting scheme:

$$\widehat{A}'(\boldsymbol{z}, h) := \widehat{D}\left(\widehat{S}'\left(\widehat{D}\left(\boldsymbol{z}, \frac{h}{2}\right), h\right), \frac{h}{2}\right), \tag{46}$$

Remark

Automatically, any strongly potential second order weak scheme will be of truly second order weak convergence in QRHM.

$\rightsquigarrow$ Following the same strategy in [BB23a], we can obtain a theoretically second order scheme for (S, V):

$$\widehat{\boldsymbol{Z}}_h = \widehat{A}'(\boldsymbol{z}, h), \qquad \widehat{\boldsymbol{Y}}_h^{(1)} = \boldsymbol{y}^{(1)} + (\boldsymbol{z} + \widehat{\boldsymbol{Z}}_h)\frac{h}{2}, \qquad \widehat{Y}_h^{(2)} = y^{(2)} + (z^2 + (w^\top \cdot \widehat{\boldsymbol{Z}}_h)^2)\frac{h}{2},$$

$$\widehat{S}_h = s \exp\left((-\frac{\left(x_1 z_0^{(1)} + \theta\right)}{\eta} - \frac{1}{2}(ab^2 + c))h + (ab + \frac{1}{\eta})(\widehat{Y}_h^{(1)} - Y_0^{(1)}) \right.$$
$$\left. + \frac{x_1}{\eta}(\widehat{Y}_h^{(1,1)} - Y_0^{(1,1)}) - \frac{a}{2}(\widehat{Y}_h^{(2)} - Y_0^{(2)}) + \frac{\widehat{Z}_h^{(1)} - Z_0^{(1)}}{\eta} \right).$$

Numerics: Benchmarks

Figure: Sample paths of S_t and V_t in QRHM with $H = 0.1$ and $N = 3$, the number of time steps is $M = 256$.

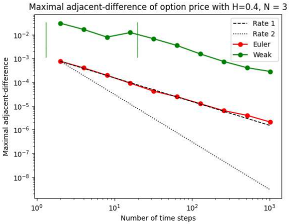

Figure: Maximal adjacent differences of European call option prices with $H = 0.4$ under $N = 3$.

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