Stationary Processes

1) Definitions:

① Def: i) Autocovariance func. $\gamma_X(\cdot,\cdot)$ of s.p. $(X_t)_{t\in T}$, s.t. $Var(X_t) < \infty$, $\forall t$ is:

$\gamma_X(r,s) = cov(X_r, X_s)$, for $s, t \in T$.

ii) $(X_t)_{t\in T}$ is (weakly) stationary if:

(a) $\mathbb{E}(|X_t|^2) < \infty$, $\forall t$ (b) $\mathbb{E}(X_t) = m$, $\forall t$.

(c) $\gamma_X(r+h, s+h) = \gamma_X(r,s)$, $\forall h$

Rmk: Note that in the case, $\gamma_X(0,h)$

$\equiv Cov(X_t, X_{t+h})$, $\forall t$

$\Rightarrow$ We can set $\gamma_X(t-s) = \gamma_X(s,t)$.

Define autocorrelation func. (a.c.f)

by: $\rho_X(h) := \gamma_X(h)/\gamma_X(0) = Corr(X_t, X_{t+h})$

iii) $(X_t)_{t\in T}$ is strictly stationary if:

$(X_{t_1}, X_{t_2} \cdots X_{t_n}) \overset{d}{\sim} (X_{t_1+h}, \cdots X_{t_n+h})$, $\forall t_i, h$.

Rmk: Strictly stat. + Finite 2^{nd} moment

$\Rightarrow$ weakly stationary.

③ Property of autocov. func. :

Next, we assume $(X_t)_{t\in T}$ is stationary, equip with $\gamma_X(\cdot)$, the autocov. func.

Prop. i) $\gamma(0) \geq 0$. ii) $|\gamma(h)| \leq \gamma(0)$

iii) $\gamma(h) = \gamma(-h)$. $\forall h \in T$.

Thm. $\gamma(\cdot)$ defined on T is autocov. func. of stationary time series $\Leftrightarrow$ it's even and nonnegative definite.

Pf: By Kolmogorov extension Thm.

Rmk: a.c.f $\rho_X(\cdot)$ satisfies all the properties above with $\rho_X(0) = 1$.

Def: $\{x_k\}_1^n$ is observation of (X_t). The sample autocov. func. of $\{x_k\}$ is

$$\hat{\gamma}(h) \triangleq \frac{1}{n} \sum_{j=1}^{n-h} (x_{j+h} - \bar{x})(x_j - \bar{x}). \text{ and}$$

set $\hat{\gamma}(-h) = \hat{\gamma}(h)$. $\forall h \in \mathbb{Z}^+$.

Rmk: The divisor $1/n$ is to ensure $\hat{\Gamma}_n = (\hat{\gamma}(i-j))$ is nonnegative definite.

(2) Estimation of components:

Consider the classical decomposition model:

$X_t = m_t + s_t + Y_t$.

i) m_t is trend component

ii) s_t is seasonal component with period $= d$.

iii) Y_t is random white noise.

We expect to extract and estimate m_t, s_t and the residual Y_t is stationary.

① Absence of Seasonality:

Consider $X_t = m_t + Y_t$. $t = 1, 2, \cdots n$.

$\mathbb{E}(Y_t) = 0$. We want to estimate m_t:

i) LSE: If $m_t = \sum_0^n a_k t^k$. Then we find

$\hat{m}_t = \arg\min \sum_t (X_t - m_t)^2$.

ii) Mean: For $q \in \mathbb{Z}^+$. if m_t is approxi. linear on $[t-q, t+q]$.

Note $W_t = (2q+1)^{-1} \left(\sum_{-q}^{q} (m_{t+j} + Y_{t+j}) \right)$

$\simeq m_t$. $q+1 \le t \le n-q$.

We set $\hat{m}_t = (2q+1)^{-1} \sum_{-q}^{q} X_t$.

For unobserved value, set $\begin{cases} X_t = X_1, & t<1 \\ X_t = X_n, & t>n \end{cases}$

iii) <u>Differentiating:</u>

Set $BX_t = X_{t-1}$, $\nabla X_t = (1-B)X_t$.

Note if $m_t = \sum_{0}^{n} a_k t^k$, $\mathbb{E}(Y_t) \equiv 0$.

Then: $\nabla^n X_t = a_n \cdot n! + \nabla^n Y_t$.

which's a stationary sequence.

② <u>General model:</u>

Consider $X_t = m_t + s_t + Y_t$ where

$\mathbb{E}(Y_t) = 0$, $s_{t+d} = s_t$, $\sum_{1}^{d} s_j = 0$.

i) <u>Small trend:</u>

If m_t is small $\Rightarrow$ consider it's const.

set: $\begin{cases} \hat{m}_t = \frac{1}{d} \sum_{1}^{d} X_j \\ \hat{s}_t = \frac{2}{d} \sum_{1}^{d/2} (X_{t+j} - \hat{m}_j) \end{cases}$

ii) <u>Average estimation:</u>

set $\tilde{m}_t = \begin{cases} \frac{1}{d}\left(\frac{1}{2}(X_{t-q} + X_{t+q}) + \sum_{t-q+1}^{t+q-1} X_k\right), & d = 2q \\ \frac{1}{d} \sum_{t-q}^{t+q} X_k, & d = 2q+1 \end{cases}$

$w_k = \text{deviation}\,\{(X_{k+jd} - \tilde{m}_{k+jd})\}_{j:\ q < k+jd \le n-q}$

$\hat{s}_k = w_k - \frac{1}{d} \sum_{1}^{d} w_i$.

Then set $r_i = X_i - \hat{s}_i$ to remove the seasonality. reduced to ①.

iii) <u>Differencing at lag d:</u>

Def: $\nabla_d X_t = (1 - B^d) X_t$.

$\Rightarrow \nabla_d X_t = m_t - m_{t-d} + Y_t - Y_{t-d}$.

So we eliminate seasonality.

$\Rightarrow$ It reduces to ①.

<u>Rmk:</u> We can use program PEST to perform all the methods above.