ARMA Processes.

(1) Causal and invertible:

① Def. i) Set $Z_t \sim WN(0, \sigma^2)$. white noise if it has cov. func. $\gamma(h) = \sigma^2 \delta(h)$.

ii) For $(X_t)_{t \in \mathbb{Z}}$. it's ARMA(p, q) - process if : $\exists\, \theta_i, \phi_i \in \mathbb{R}'$. s.t.

(a) X_t is stationary seq.

(b). $X_t - X_{t-1} \cdot \phi_1 \cdots - X_{t-p} \phi_p = Z_t + Z_{t-1} \cdot \theta_1 + \cdots + Z_{t-q} \cdot \theta_q$. $\forall t \in \mathbb{Z}$.

where $Z_t \sim WN(0, \sigma^2)$.

Rmk: Set $\phi(z) = 1 - \phi_1 z \cdots - \phi_p z^p$.

$\theta(z) = 1 + \theta_1 z \cdots + \theta_q z^q$.

$\Rightarrow \phi(B) X_t = \theta(B) Z_t$.

We call $\phi(B) X_t = Z_t$ by AR(p). process and $X_t = \theta(B) Z_t$ by MA(q).

iii) For ARMA(p.q) - process $\phi(B) X_t = \theta(B) Z_t$ it's causal w.r.t (Z_t) if $\exists (\psi_i) \subset \mathbb{R}'$. s.t. $X_t = \sum_{i \geq 0} \psi_i Z_{t-i}$. $\sum_{i \geq 0} |\psi_i| < \infty$.

Rmk: It's a relation between X_t and Z_t

Lemma. For (X_t). seq. of r.v.'s. If:

i) $\sup_t \mathbb{E}|X_t| < \infty$ and ii) $\sum_{\mathbb{Z}} |\psi_i| < \infty$

Then: $\psi(B)X_t = \sum_{\mathbb{Z}} \psi_i X_{t-i}$ converges absolutely. a.s.

if additionally. $\sup_t \mathbb{E}|X_t|^2 < \infty$. then it also converges absolutely in L^2.

Pf: 1) Note $\mathbb{E}(\sum |\psi_i||X_{t-i}|) < \infty$.

2) Check Cauchy criterion: $\mathbb{E}(\sum_n^m |\psi_i X_{t-i}|)^2 \to 0$ (as. $n.m \to \infty$).

prop. If (X_t) is stationary seq. with cov. func. $\gamma(\cdot)$. and $\sum |\psi_i| < \infty$. Then:

$Y_t = \psi(B)X_t$ exists. a.s. and in L^2. is a stationary seq with cov. $\gamma_Y(h) = \sum_{j,k \in \mathbb{Z}} \psi_j \psi_k \gamma(h-j+k)$

Pf. By Lemma, Y_t is well-def.

And $\mathbb{E}(Y_t Y_{t+h}) = \sum \psi_j \psi_k (\gamma(h-j+k) + (\mathbb{E}X_t)^2)$.

Thm. (X_t) is ARMA$(p.q)$. st. $Z(\theta(z)) \cap Z(\phi(z)) = \varnothing$.

Then (X_t) is causal $\Leftrightarrow$ $\phi(z) \neq 0$ on $\{|z| \leq 1\}^c$.

Additionally. (ψ_i) is determined by:

$\psi(z) = \sum_{i \geq 0} \psi_i z^i = \theta(z)/\phi(z)$. $|z| \leq 1$.

Pf. 1) Set $\xi(z) = 1/\phi(z) = \sum_{i \geq 0} \xi_i z^i$. $|z| \leq 1+\varepsilon$.

for some $\varepsilon > 0$. if $\phi(z) \neq 0$ on $\{|z|=1\}$.

$\Rightarrow X_t = \xi(B)\theta(B)Z_t = \psi(B)Z_t$.

ii) Conversely. Note $\theta(B)Z_t = \phi(B)X_t$

$= \phi(B)\psi(B)Z_t$

$\Rightarrow$ take inner product with Z_{t-k}.

So: $\theta(z) = \phi(z)\psi(z)$.

Rmk: It's easy to see $X_t = \psi(B)Z_t$

is unique stationary solution of

ARMA(p,q) - equation $\phi(B)X_t = \theta(B)Z_t$.

Cor. i) If $Z(\phi(z)) \cap Z(\theta(z)) = \emptyset$.

and $\phi(z) = 0$ for some $|z|=1$.

Then there's no stationary

solution for $\phi(B)X_t = \theta(B)Z_t$.

ii) If $M = Z(\phi(z)) \cap Z(\theta(z))$. Then:

$M \cap \{|z|=1\} \overset{\text{or}}{=} \emptyset \Rightarrow X_t = \psi(B)Z_t$.

is the unique stat. solution.

$M \cap \{|z|=1\} \overset{\text{or}}{\neq} \emptyset \Rightarrow \phi(B)X_t = \theta(B)Z_t$

has more than one stat. solution.

Def: For ARMA(p,q) - process $\phi(B)X_t = \theta(B)Z_t$. it's

invertible w.r.t. (X_t) if $\exists\ (\pi_i) \subset \mathbb{R}$. s.t.

$Z_t = \sum_{i\geq 0} \pi_i X_{t-i}$. $\sum_{i\geq 0} |\pi_i| < \infty$.

Rmk: Thm above also works on invertibility by symmetry.

Cor. If $\varphi(z)\theta(z) \neq 0$ on $\{|z| \leq 1\}$ ($\in \mathbb{C}$).

Then $X_t = \psi(B) Z_t$, $Z_t = \pi(B) X_t$.

where $\psi(z) = \theta(z)/\varphi(z)$, $\pi(z) = \psi^{-1}(z)$.

Cor. If $\varphi \neq 0$ on $\{z \in \mathbb{C} \mid |z| = 1\}$. Then:

$\varphi(B) X_t = \theta(B) Z_t$ has a unique stat. solution $X_t = \sum_{j \geq 0} \psi_j Z_{t-j}$. $\psi(z) = \theta(z)/\varphi(z)$.

Pf: Note $\exists \delta > 1$, s.t. $1/\varphi(z) = \xi(z)$ will absolutely converge on $\{1/\delta < |z| < \delta\}$ ($\in \mathbb{C}$).

③ Def: $(Z_t)_{\mathbb{Z}} \sim WN(0, \sigma^2)$. $(X_t)_{\mathbb{Z}}$ is $MA(\infty)$ w.r.t Z_t if $\exists (\psi_j) \subset \mathbb{R}$, s.t. $\sum |\psi_j| < \infty$ and

$X_t = \sum_{j \geq 0} \psi_j Z_{t-j}$. $\forall t$.

Rmk: (X_t) is stationary and mean-zero with $\gamma_X(h) = \sum_{k \geq 0} \sigma^2 \psi_k \psi_{k+|h|}$.

prop. (Criteria of $MA(q)$)

If (X_t) is zero-mean with $\gamma(\cdot)$, s.t. $\exists q \in \mathbb{Z}$, $\gamma(q) \neq 0$, $\gamma(h) = 0$, $\forall |h| > q$. Then X_t is $MA(q)$.

Pf. Set $\mathcal{M}_t = \overline{LS}\{X_s\}_{s\le t}$.

$Z_t = X_t - P_{\mathcal{M}_{t-1}} X_t$. (*).

Note $\|Z_t\| = \lim_{n\to\infty} \| X_t - P_{LS\{X_s\}_{-n\le s\le t}} X_t \|$

$\overset{\text{stat}}{=} \lim_{n\to\infty} \| X_{t+1} - P \square \| = \| Z_{t+1} \|$

Besides, by (*):

$\mathcal{M}_t = \overline{LS}\{\mathcal{M}_{t-1}, Z_t\} = \cdots = \overline{LS}\{\mathcal{M}_{t-q}, Z_t \sim Z_{t-q+1}\}$.

By $\gamma(h) = 0$ for $\forall |h| > q$.

We have $X_t = P_{\mathcal{M}_{t-q}} X_t + P_{\{Z_t \sim Z_{t-q}\}} X_t$

$= P_{\overline{LS}\{Z_{t-q} \sim Z_t\}} X_t$

Cor. Y_t is ARMA(p,q). If X_t has a same cov. func. as (Y_t). Then: (X_t) is also ARMA(p,q).

Pf. Set $W_t = \phi(B) X_t$, where $\phi(B) Y_t = \theta(B) Z_t$. $Z_t \sim WN(0, \sigma^2)$

$\Rightarrow$ (W_t) will satisfy the condition above (stat. and $\gamma(\cdot)$)

(2) Computation of Autocov.:

① Find $\psi(z) = \theta(z)/\phi(z) \Rightarrow \gamma(h) = \sigma^2 \sum_{j\ge 0} \psi_j \psi_{j+|h|}$

② Consider causal ARMA(p, q): $\phi(B) X_t = \theta(B) Z_t$.

Take inner product with X_{t-k}:

$\Rightarrow$ $\gamma(k) - \phi_1 \gamma(k-1) - \cdots - \phi_p \gamma(k-p) =$

$$\begin{cases} \sigma^2 \sum_{k \le j \le q} \theta_j \psi_{j-k}, & 0 \le k \le \max\{p, q+1\} \\ 0, & \text{otherwise.} \end{cases}$$

Rmk: $\gamma(\cdot)$ has form: $\gamma(h) = \sum_{i=1}^{k} \sum_{j=0}^{r_i - 1} \beta_{ij} h^j \xi_i^{-h}$

for $\forall h \ge p \wedge (q+1) - p$. $Z(\phi) = \{\xi_i\}_{i=1, j=0}^{k, r_i}$

Cor. For ARMA process with $\phi(z) \ne 0$ on $|z| = 1$.

Then: $\exists c > 0$, $s \in (0,1)$. $|\gamma(h)| \le c s^{|h|}$.

So, $\sum_{\mathbb{Z}} |\gamma(h)| < \infty$.

(3) Autocov. g.f:

Def: For stationary process $(X_t)_{\mathbb{Z}}$ with autocov. $\gamma(\cdot)$, the autocov. generating func. is $G(z) = \sum_{\mathbb{Z}} \gamma(h) z^h$. St. $\{r < |z| < r^{-1}\}$ is its convergence domain.

Rmk: i) If $X_t = \sum_{\mathbb{Z}} \psi_i Z_{t-i}$, where $\sum_{\mathbb{Z}} |\psi_i| |z^i| < \infty$ on $\{r < |z| < r^{-1}\}$, $\exists r < 1$. and

$Z_t \sim WN(0, \sigma^2)$.

Recall $\gamma(h) = \sigma^2 \sum_k \psi_k \psi_{k+|h|}$

$\Rightarrow$ $G(z) = \sigma^2 \psi(z) \psi^{-1}(z)$.

ii) For ARMA(p,q), we have:

$\Rightarrow$ $G(z) = \sigma^2 \theta(z)\theta(z^{-1}) / \varphi(z)\varphi(z^{-1})$.

Prop. (X_t) is ARMA(p,q). $\phi(B)X_t = \theta(B)Z_t$.

s.t. $\phi(z)\theta(z) \neq 0$ on $|z|=1$. Then $\exists$ $\tilde{\phi}$, $\tilde{\theta}$ polynomials with degree p, q resp. nonzero on $|z|=1$. and $\exists$ $(Z_t^*) \sim WN(0, \tilde{\sigma}^2)$

s.t. $\tilde{\phi}(B)X_t = \tilde{\theta}(B)Z_t^*$.

Rmk: It means $\exists$ (Z_t^*) white noise.

s.t. X_t is MA of Z_t^*.

Pf: Suppose $\{a_i\}_{r=1}^{p} = Z(\phi(z)) \cap \{|z|<1\}$

and $\{b_i\}_{s=1}^{q} = Z(\theta(z)) \cap \{|z|<1\}$

$$\tilde{\phi}(z) = \phi(z) \prod_{r=1}^{p} \frac{(1-a_i z)}{(1-a_i^{-1} z)}$$

$$\tilde{\theta}(z) = \theta(z) \prod_{s=1}^{p} \frac{(1-b_i z)}{(1-b_i^{-1} z)}$$

Set $Z_t^* = \tilde{\theta}(B) / \tilde{\phi}(B) \cdot X_t$.

By Rmk above. it has const. a.g.f

$G_Z(z) = \tilde{\sigma}^2 (\prod_{r=1}^{p} |a_i|^2)(\prod_{s=1}^{p} |b_i|^2)$

$\Rightarrow$ (Z_t^*) is white noise.