Predictions

(1) Prediction for Stationary process:

① Assume $(X_t)_Z$ is stationary process with mean 0.

set: $\mathcal{H}_n \triangleq \overline{LS}\{X_1, X_2 \cdots X_n\}$. $\hat{X}_{n+1} = P_{\mathcal{H}_n} X_{n+1}$. $n \geq 1$.

prop. If $\gamma(0) > 0$. $\gamma(h) \xrightarrow{h \to \infty} 0$. Then $\Gamma_n = (\gamma(i-j))_{n \times n}$ of $(X_1 \cdots X_n)$ is nonsingular.

Pf: Lemma. If Σ is Var. of $(X_1 \cdots X_n)$

$$|\Sigma| = 0 \iff \exists b \in \mathbb{R}^n.\ Var(bX^T) = 0$$

By contradiction: $\exists r$. st. $|\Gamma_r| \neq 0$.

but $X_{r+1} = \sum_1^r a_i X_i$

$\Rightarrow$ By stationary: $X_{r+h+1} = \sum_1^r a_i X_{i+h}$

i.e. $\forall n \geq r+1$. $X_n = a^{(n)T} \vec{X}_r$. by iteration

where $\vec{X}_r = (X_1, \cdots X_r)$.

Note $\sum_1^r \lambda_1 (a_i^{(n)})^2 \leq \gamma(0) = cov(X_n, X_n)$

$$\leq \sum_1^r |\gamma(n-j)||a_j^{(n)}|.$$

So: $\gamma(h) \nrightarrow 0$. contradiction!

cor. Under the conditions above, we have.

$\hat{X}_{n+1} = \sum_{i=1}^{n} \hat{\phi}_{ni} X_{n+1-i}$ $\phi_n = \Gamma_n^{-1} \gamma_n$.

where $\gamma_n = (\gamma(1) \cdots \gamma(n))$, $\phi_n = (\phi_{n1} \cdots \phi_{nn})$

and the MSE $v_n = \gamma(0) - \gamma_n' \Gamma_n^{-1} \gamma_n$.

Pf: $cov(\hat{X}_{n+1}, X_{n+1-j}) = \sum \gamma(i-j)\phi_{ni}$

$= cov(X_{n+1}, X_{n+1-j})$

$= \gamma(j)$, $j \le n$.

So: $\Gamma_n \phi_n = \gamma_n$.

Cor. $P_{\mathcal{H}_n} X_{n+h} = \sum_1^n \phi_{ni}^{(h)} X_{n+1-i}$.

where $\Gamma^n \phi_n^{(h)} = \gamma_n^{(h)} = (\gamma(h) \cdots \gamma(h+n-1))$

③ Computation:

i) Durbin - Levinson Algorithm:

prop. For stationary zero-mean process $(X_t)_{\mathbb{Z}}$.

with $\gamma(0) > 0$. $\gamma(h) \xrightarrow{h\to\infty} 0$. Set $v_n = MSE(\hat{X}_{n+1})$

Then: $\phi_{11} = \gamma(1)/\gamma(0)$. $v_0 = \gamma(0)$.

$$\phi_{nn} = (\gamma(n) - \sum_1^{n-1} \phi_{n-1,j}\gamma(n-j))v_{n-1}^{-1}.$$

$$\begin{pmatrix} \phi_{n1} \\ \vdots \\ \phi_{n,n-1} \end{pmatrix} = \begin{pmatrix} \phi_{n-1,1} \\ \vdots \\ \phi_{n-1,n-1} \end{pmatrix} - \phi_{nn} \begin{pmatrix} \phi_{n-1,n-1} \\ \vdots \\ \phi_{n-1,1} \end{pmatrix}.$$

$$v_n = v_{n-1}(1 - \phi_{nn}^2).$$

Pf: Set $K_1 = \overline{LS}\{X_2 \cdots X_n\}$. $K_2 = LS\{X_1 - P_{K_1}X_1\}$.

So: $\hat{X}_{n+1} = P_{k_1} X_{n+1} + P_{k_2} X_{n+1}$.

Use Gram-Schmidt method to generate the relation of coefficients.

<u>Def</u>: The partial autocorrelation function (pacf.)

$\alpha(k) := \text{cor}(X_{k+1} - P_{\bar{s}\{1, X_2 \cdots X_k\}}, X_1 - P_{\bar{s}\{1, X_2 \cdots X_k\}})$.

<u>Rmk</u>: For $\phi(B) X_t = Z_t$. we have $\alpha(k) = 0$

for $k > \deg(\phi(z))$.

<u>Cor.</u> (Another def of pacf.)

$\alpha(n) = \phi_{nn}$. with same denotation above.

ii) <u>Innovation Algorithm</u>:

<u>prop.</u> For general zero-mean process $(X_t)_{\mathbb{Z}}$. Set

$E(X_i X_j) = k(i,j)$. s.t. $(k(i,j))_{n \times n}$ is nonsingular

for $\forall n$, Then: $\hat{X}_{n+1} = \sum_1^n \theta_{nj} (X_{n+1-j} - \hat{X}_{n+1-j})$

where $V_0 = k(1,1)$. $\theta_{n,n-k} = V_k^{-1} (k(n+1,k+1) - \sum_0^{k-1} \theta_{k,k-j} \theta_{n,n-j} V_j)$

$V_n = k(n+1,n+1) - \sum_0^{n-1} \theta_{n,n-j}^2 V_j$

<u>Pf</u>: Consider to decompose X_{n+1} into the

innovation space: $\mathcal{H}_n = \bar{s}\{ X_1 - \hat{X}_1 \cdots X_n - \hat{X}_n \}$.

by Gram-Schmidt method.

Cor. $P_n X_{n+h} = \sum_{h}^{n+h-1} \theta_{n+h-1,j} (X_{n+h-j} - \hat{X}_{n+h-j})$

Pf: $LHS = P_n \hat{X}_{n+h} = \cdots$

(2) <u>Prediction of ARMA(p,q):</u>

For $\phi(B) X_t = \theta(B) Z_t$. Set $W_t = \begin{cases} \sigma^{-1} X_t, & t \le m \\ \sigma^{-1} \phi(B) X_t, & t > m \end{cases}$ (A)

where $m = p \vee q$.

Then apply the innovations algorithm on (W_t)

to find (θ_{ni}).

<u>Rmk:</u> Note that $\theta_{nj} = 0$ if $n \ge m$, $j > q$.

it's from $k(i,j) = E(W_i W_j) = 0$ if

$i > m$, $|i-j| > q$.

$$\Rightarrow \hat{X}_{n+1} = \begin{cases} \sum_{1}^{n} \theta_{nj} (X_{n+1-j} - \hat{X}_{n+1-j}), & n < m \\ \phi_1 X_n + \cdots \phi_p X_{n+1-p} + \sum_{1}^{q} \theta_{nj} (X_{n+1-j} - \hat{X}_{n+1-j}) \end{cases}$$

<u>Rmk:</u> i) We only need to restore $p+q$ infos.

in ARMA(p,q) case.

ii) $MSE(\hat{X}_{n+1}) = MSE(\hat{W}_{n+1})$

$$\text{And } P_n X_{n+h} = \begin{cases} \sum_{h}^{n+h-1} \theta_{n+h-1,j} (W_{n+h-j} - \hat{W}_{n+h-j}), & h \le m-n \\ \sum_{1}^{p} \phi_i P_n X_{n+h-i} + \sum_{h}^{q} \theta_{n+h-1,j} (W \cdots). \end{cases}$$

follows from applying P_n on (A).

And $X_{n+h} = X_{n+h} - \hat{X}_{n+h} + \hat{X}_{n+h}$

$$= \sum_1^p \phi_i X_{n+h-i} + \sum_0^q \theta_{n+h-1,j} (X_{n+h-j} - \hat{X}_{n+h-j})$$

Subtract $P_n X_{n+h} = \cdots$ to the equation above.

We have: $\Phi \begin{pmatrix} X_{n+1} - P_n X_{n+1} \\ \vdots \\ X_{n+h} - P_n X_{n+h} \end{pmatrix} = \Theta \begin{pmatrix} X_{n+1} - \hat{X}_{n+1} \\ \vdots \\ X_{n+h} - \hat{X}_{n+h} \end{pmatrix}$

where $\Phi = -(\phi_{i-j})_{h\times h}$, $\Theta = (\theta_{n+i-1,i-j})_{h\times h}$.

Rmk: Φ and Θ are both lower triangular.

Thm: For causal invertible ARMA(p,q). Set $\tilde{X}_t := P_{\overline{sp}\{X_j,\, j\le n\}} X_t$. Then we have:

$$\tilde{X}_{n+h} = -\sum_1^{\infty} \pi_i \tilde{X}_{n+h-i} \quad \text{and} \quad \tilde{X}_{n+h} = \sum_h^{\infty} \psi_i Z_{n+h-i}.$$

where $\pi(z) = \phi(z)/\theta(z)$, $\psi(z) = \pi(z)^{-1}$.

Besides, $\mathbb{E}((X_{n+h} - \tilde{X}_{n+h})^2) = \sigma^2 \sum_0^{h-1} \psi_i^2$.

Pf: Note: $Z_{n+h} = X_{n+h} + \sum_1^{\infty} \pi_j X_{n+h-j}$.

$$X_{n+h} = \sum_1^{\infty} \psi_i Z_{n+h-i}$$

Apply $P_{\overline{sp}\{X_j,\, j\le n\}}$ on both sides.

Rmk: i) We can use the relation above to generate $(\tilde{X}_t)$.

ii) $(\tilde{X}_{n+k} - X_{n+k})$ isn't uncorrelated:

$$\mathbb{E}((X_{n+k} - \tilde{X}_{n+k})(X_{n+h} - \tilde{X}_{n+h})) \overset{k\ge h}{=} \sigma^2 \sum_0^{h-1} \psi_i \psi_{k+i-h}$$