# Estimations

(1) $\mu$ and $\gamma(\cdot)$:

**Prop.** If $(X_t)$ is stationary with mean $\mu$ and autocov. $\gamma(\cdot)$. Then:

$$\gamma(n) \xrightarrow{n\to\infty} 0 \Rightarrow Var(\bar{X}_n) \xrightarrow{n\to\infty} 0.$$

Additionally, $\sum_{\mathbb{Z}} |\gamma(n)| < \infty \Rightarrow Var(\bar{X}_n) \sim \frac{\sum_{\mathbb{Z}} \gamma(k)}{n}$

**Pf.**
$$Var(\bar{X}_n) \overset{i)}{=} \frac{1}{n} \sum_{|h|<n} \left(1 - \frac{|h|}{n}\right) \gamma(h)$$
$$\overset{ii)}{\leq} \frac{1}{n} \sum_{|h|<n} \gamma(h).$$

Apply Stolz on ii) to obtain the first conclusion. the latter is from i).

**Cor.** $(X_t)_{\mathbb{Z}}$ is stationary seq. s.t.

$X_t = \mu + \sum_{\mathbb{Z}} \psi_j Z_{t-j}$, $Z_t \sim IID(0, \sigma^2)$

If $\sum_{\mathbb{Z}} |\psi_i| < \infty$. Then:

$\bar{X}_n \sim AN(\mu, \sum_{\mathbb{Z}} \gamma(h) / n)$

**Thm.** Under the conditions of cor. above.

If i) $\mathbb{E}(Z_t^4) < \infty$ or ii) $\sum_{\mathbb{Z}} \psi_j^2 |j| < \infty$

Then: $\hat{\rho}(h) \sim AN(\rho(h), W/n)$.

where $\hat{\vec{\ell}}(h) = (\hat{\ell}(1) \cdots \hat{\ell}(h))$. $\hat{\ell}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$.

$$W_{ij} = \sum_{k=1}^{\infty} (\ell(k+i) + \ell(k-i) - 2\ell(i)\ell(k))(\ell(k+j) + \ell(k-j) - 2\ell(j)\ell(k)).$$

(2) ARMA Models:

Next, we will model $p, q, \vec{\phi}, \vec{\theta}$ and $\sigma^2$.

① AR(p)-process:

i) Consider zero-mean causal AR(p) model:

$\phi(B) X_t = Z_t$, $Z_t \sim WN(0, \sigma^2)$.

(By causality: $X_t = Z_t + \sum_{j=1} \psi_j Z_{t-j}$.)

product with $X_{t-j}$, $j = 0, \cdots p$. We have:

$$\begin{cases} \sigma^2 = \gamma(0) - \phi^T \gamma_p & j = 0 \\ \Gamma_p \phi = \gamma_p & j > 0 \end{cases}$$ by orthogonality.

We have Yule-Walker estimator:

$$\begin{cases} \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}^T \hat{\gamma}_p \\ \hat{\Gamma}_p \hat{\phi} = \hat{\gamma}_p . \end{cases}$$ $\hat{\Gamma}_p$, $\hat{\gamma}_p$ are estimated by data $(X_i)_1^n$.

Thm i) $\hat{\phi}$ and $\hat{\gamma}$ also satisfy:

$$\hat{\gamma}(h) - \hat{\phi}_1 \hat{\gamma}(h-1) \cdots - \hat{\phi}_p \hat{\gamma}(-p+h) = \begin{cases} 0 & h = 1 \cdots p \\ \hat{\sigma}^2 & h = 0 \end{cases}$$

ii) $n^{\frac{1}{2}}(\hat{\phi} - \phi) \xrightarrow{d} N(0, \sigma^2 \Gamma_p^{-1})$. $\hat{\sigma}^2 \xrightarrow{P} \sigma^2$.

ii) $p$ is commonly unkown.

consider for $m$. use $\hat{\phi}_m = (\hat{\phi}_{m1} \cdots \hat{\phi}_{mm})$

$= \hat{R}_m^{-1} \hat{\rho}_m$. follows from the relation above.

$\Rightarrow X_t - \hat{\phi}_{m1} X_{t-1} \cdots - \hat{\phi}_{mm} X_{t-m} = Z_t$. where

$Z_t \sim WN(0, \hat{\gamma}(0) - \hat{\rho}_m^T \hat{\gamma}_p) \triangleq WN(0, V_m)$

<u>Rmk:</u> We can also use the Durbin-Levinson algorithm to generate $(\hat{\phi}_{mi})_{i \le m}$. if $\hat{\gamma}(0) > 0$.

(It also satisfies the relation equations)

<u>Thm.</u> $n^{\frac{1}{2}}(\hat{\phi}_m - \phi_m) \xrightarrow{d} N(0, \sigma^2 \Gamma_m^{-1})$.

② MA(q)-process:

consider for $m$. $\hat{\theta}_m = (\hat{\theta}_{m1} \cdots \hat{\theta}_{mm})$.

$X_t = Z_t + \hat{\theta}_{m1} Z_{t-1} + \cdots \hat{\theta}_{mm} Z_{t-m}$. $Z_t \sim WN(0, \hat{V}_m)$

We can apply the innovation algorithm to obtain the estimators:

$$\hat{\theta}_{m, m-k} = \hat{V}_k^{-1} (\hat{\gamma}(m-k) - \sum_0^{k-1} \hat{\theta}_{m, m-j} \hat{\theta}_{k, k-j} \hat{V}_j). \quad k = 0 \sim m-1.$$

$$\hat{V}_0 = \hat{\gamma}(0) - \sum_0^{m-1} \hat{\theta}_{m, m-j}^2 \hat{V}_j.$$

Thm. For causal invertible ARMA(p,q)-process

$\phi(B)X_t = \theta(B)Z_t$. $Z_t \sim IID(0, \sigma^2)$. $\mathbb{E}(Z_t^4) < \infty$.

$\psi(z) = \theta(z)/\phi(z)$. Then: $\forall\ (m(n))_{n \geq 1}$ s.t.

$m(n) = o(n^{\frac{1}{3}})$. $(n \to \infty)$. we have:

$$n^{\frac{1}{2}}(\hat{\theta}_{m1} - \psi_1, \cdots \hat{\theta}_{mk} - \psi_k)^T \xrightarrow{d} N(0, A)$$

$$\hat{V}_m^2 \xrightarrow{P} \sigma^2. \quad \text{where } a_{ij} = \sum_1^{i \wedge j} \psi_{i-k}\psi_{j-k}.$$

Rmk: For MA(q)-process. $\hat{\theta}_m$ isn't consistent with $\theta$ (only for special $(m(n))$).

③ ARMA(p,q):

For causal ARMA(p,q)-process:

$\phi(B)X_t = \theta(B)Z_t$. $Z_t \sim WN(0, \sigma^2)$.

$\Rightarrow X_t = \psi(B)Z_t$. $\psi(z) = \theta(z)/\phi(z)$.

where $\psi_0 = 1$. $\psi_j = \theta_j + \sum^{j \wedge p} \phi_i \psi_{j-i}$. $\theta_j = 0\ \ j > q$. $\phi_j = 0\ \ j > p$.

By Thm. above. replace $\psi_i$ by $\hat{\theta}_{m,i}$ (approxi.).

$$\Rightarrow \hat{\theta}_{m,j} = \theta_j + \sum_1^{j \wedge p} \phi_i \hat{\theta}_{m,j-i}. \quad j = 1, \cdots p+q.$$

From the equation $j = q+1, \cdots p+q$:

We have:

$$\begin{pmatrix} \hat{\theta}_{m,q+1} \\ \vdots \\ \hat{\theta}_{m,p+q} \end{pmatrix} = (\hat{\theta}_{m,q+i-j})_{p \times p} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix}$$

S. obtain estimate $\hat{\phi}$. replace into equation $j = 1 \sim q$.

to get estimate $\hat{\theta}$.