Spectral Decomposition

Next, we consider complex-valued process (X_t). And Set: $\langle X, Y\rangle = \mathbb{E}(X\bar{Y})$. $\gamma(x,\eta) \hat{=} \langle X_x, X_\eta\rangle - \mathbb{E}(X_x)\mathbb{E}(\bar{X}_\eta)$.

(1) Preliminary:

Thm (Herglotz).

$\gamma: \mathbb{Z} \to \mathbb{C}$ is nonnegative definite

$\Leftrightarrow$ $\gamma(h) = \int_{(-\pi,\pi]} e^{ihv}\, dF(v)$ where F

is right-conti. bdd. $\nearrow$. $F(-\pi) = 0$.

Def: F is called spectral d.f. of $\gamma(\cdot)$. And if $F(\lambda) = \int_{-\pi}^{\lambda} f(v)dv$.

$\Rightarrow$ we call f is spectral density

Pf: ($\Rightarrow$). Set $f_N(v) = \frac{1}{2\pi N} \sum_{r,s=1}^{N} e^{-irv}\gamma(r-s)e^{-isv}$.

$= \frac{1}{2\pi N} \sum_{|m|<N} e^{-imv}(N-|m|)\gamma(m)$. ≥ 0.

$F_N(v) = \int_{-\pi}^{v} f_N(r)dr \Rightarrow F_N(\pi) = \gamma(0)$.

Besides, $\int_{-\pi}^{\pi} e^{ihv} dF_N(v) = \begin{cases} 0 & h \geq |N| \\ (1-\frac{|h|}{N})\gamma(h). & \end{cases}$

By Helly's Selection:

Since (F_N) is bdd. d.f.

$\exists (F_{N_k})$. s.t. $F_{N_k} \xrightarrow{v} F$. F is d.f.

So: let $N_k \to \infty$. $\gamma(h) = \int_{-\pi}^{\pi} e^{ihv} dF(v)$.

($\Leftarrow$). easy to check $\gamma(h)$ is Hermitian and it's nonnegative definite.

Cor. $\gamma(h)$ is autocov. func of stationary seq $(X_t)_{\mathbb{Z}}$ $\Leftrightarrow$ $\gamma(h) = \int_{-\pi}^{\pi} e^{ihv} dF(v)$. where F is right-conti. bdd. $\uparrow$.

Cor. If $\gamma(\cdot): \mathbb{Z} \to \mathbb{C}$. satisfies $\sum_{\mathbb{Z}} |\gamma(h)| < \infty$

Then: γ is autocov. of a stationary process $\Leftrightarrow$ $f(\lambda) := \sum_{\mathbb{Z}} e^{-in\lambda} \gamma(n)/2\pi \geq 0$

Pf: ($\Rightarrow$). By Pf of Thm above.

$$f_N(v) \xrightarrow{N\to\infty} f(v). \Rightarrow f(v) \geq 0$$

($\Leftarrow$). We have $\gamma(h) = \int_{-\pi}^{\pi} e^{ihv} f(v) dv$.

Set $F(\lambda) = \int_{-\pi}^{\lambda} f(v) dv$. $\uparrow$.

Satisfies the conditions above!

Rmk: If (X_t) is real-valued stationary.

Then: $f_X(\lambda) = f_X(-\lambda)$. $F_X(\lambda) + F_X(-\lambda^-) = F_X(\pi^-)$.

$$\gamma_X(h) = \int_{-\pi}^{\pi} \cos vh \, dF(v).$$

Pf: Note $\gamma_X(h)$ is real. value.

(2) ARMA(p,q):

① Lemma. If $\{Y_t\}$ is stationary, 0-mean, with spectral d.f. $F_Y(\cdot)$, $X_t = \sum_{\mathbb{Z}} \psi_j Y_{t-j}$, $\sum|\psi_j| < \infty$.

Then: X_t is stationary with spectral d.f. $F_X(\cdot) = \int_{-\pi}^{\cdot} |\sum_{j\in\mathbb{Z}} \psi_j e^{-ij\nu}|^2 \, dF_Y(\nu)$.

Pf: Recall $\gamma_X(h) = \sum_{j,k\in\mathbb{Z}} \psi_j \bar{\psi}_k \gamma_Y(h-j+k)$

$= \int_{-\pi}^{\pi} e^{ih\nu} |\sum \psi_j e^{-ij\nu}|^2 dF_Y(\nu)$.

Cor. $f_X(\lambda) = |\psi(e^{-\lambda i})|^2 f_Y(\lambda)$.

Thm. (X_t) is ARMA(p,q), s.t. $\phi(z)\theta(z) \neq 0$ on $\{z \in \mathbb{C} \mid |z|=1\}$. Then: $F_X(\lambda) = \frac{\sigma^2}{2\pi} \cdot \frac{|\theta(e^{i\lambda})|^2}{|\phi(e^{i\lambda})|^2}$.

Pf: Note $\gamma(0) = \sigma^2 = \int_{-\pi}^{\pi} f(\lambda) d\lambda$.

$\gamma(h) = 0$, if $h \neq 0$.

$\Rightarrow f \equiv \text{const.} \Rightarrow f(\lambda) = \sigma^2/2\pi$.

Thm. For ARMA(p,q) $\phi(B)X_t = \theta(B)Z_t$, s.t. $\phi(z)\theta(z) \neq 0$ on $|z|=1$. We have $\tilde{\phi}(B)\tilde{X}_t = \tilde{\theta}(B)\tilde{Z}_t$ is causal and invertible ARMA(p,q), has same spectral density as (X_t).

Pf: Similar as before. Consider:

$$\tilde{\phi}(z) = \prod_{1}^{j} (1-\alpha_i^{-1} z) \prod_{j+1}^{p} (1-\bar{\alpha}_i z)$$

$$\tilde{\theta}(z) = \prod_{1}^{i} (1-b_k^{-1} z) \prod_{i+1}^{q} (1-\bar{b}_k z)$$

(drop the zeros inside $|z|<1$:

$(\alpha_i)_{j+1}^{p}$ and $(b_k)_{i+1}^{q}$.)

Set $\tilde{Z}_t \sim N(0, \sigma^2 (\prod_{j+1}^{p} |\alpha_k|)^2 (\prod_{i+1}^{q} |b_k|)^2)$

$\Rightarrow \tilde{\phi}(B)\tilde{X}_t := \tilde{\theta}(B)\tilde{Z}_t$.

$$f_{\tilde{X}}(\lambda) = \frac{|\tilde{\theta}(e^{-i\lambda})|^2}{|\tilde{\phi}(e^{-i\lambda})|^2} \cdot \frac{\sigma^2 (\prod_{j+1}^{p} |\alpha_k|)^2}{2\pi (\prod_{i+1}^{q} |b_k|)^2} = f_X(\lambda).$$

Cor. $\exists Z_t^* \sim \tilde{Z}_t$ s.t. $\tilde{\phi}(B) X_t = \tilde{\theta}(B) Z_t^*$.

Prop. For $\phi(B) X_t = \theta(B) Z_t$, $Z_t \sim WN(0, \sigma^2)$. If $Z(\phi) \cap Z(\theta) = \emptyset$, $\phi \neq 0$ on $|z|=1$, $\theta \neq 0$ on $|z|<1$. Then: $Z_t \in \overline{LS}\{X_s, s \le t\}$.

Pf: Set $\theta(z) = \theta_1(z)\theta_2(z)$

$$= \prod_{1}^{s} (1-b_i^{-1} z) \prod_{s+1}^{q} (1-b_i^{-1} z).$$

where $|b_i| > 1$, $\forall i \le s$. $|b_i| = 1$, $\forall i \ge s+1$.

$\Rightarrow \phi(B) X_t = \theta_1(B)\theta_2(B) Z_t \stackrel{\Delta}{=} \theta_1(B) Y_t$.

So: $Y_t = \phi(B) X_t / \theta_1(B)$. $\overline{LS}\{Y_s, s \le t\} \subseteq \overline{LS}\{X_s, s \le t\}$.

For $Y_t = \theta_2(B) Z_t$.

Note $\exists q_0 = q-s$. s.t. $\forall h > q_0$. $\gamma_Y(h) = 0$.

So: $Y_t = U_t + \alpha_1 U_{t-1} + \cdots \alpha_{q-s} U_{t-q+s}$. $U_t \sim WN(0, \sigma_U^2)$

where $u_t = Y_t - P_{\overline{sp}\{Y_s,\, s<t\}} Y_t$

we have: $\frac{\sigma_u^2}{2\pi} |\xi(e^{-i\lambda})|^2 = f_Y(\lambda) = \frac{\sigma^2}{2\pi} |\theta(e^{-i\lambda})|^2$.

Set $\lambda = 0 \Rightarrow \sigma_u^2 = \sigma^2$.

With $Z(\theta) \cap Z(\phi) = \emptyset \Rightarrow \xi = \theta$.

So: $(u_t, Y_t, \cdots Y_{t-n})^T$ has same cov. func. with $(Z_t, Y_t, \cdots Y_{t-n})^T$

$$\Rightarrow P_{\overline{sp}\{Y_s,\, s\le t\}} u_t = P_{\overline{sp}\{Y_s,\, s\le t\}} Z_t$$

$$\begin{array}{c} \parallel \\ u_t \in \overline{sp}\{Y_s,\, s\le t\} \end{array}$$

So: $\mathbb{E}(Z_t - P_{\overline{sp}\{Y_s,\, s\le t\}} Z_t)^2 =$

$\mathbb{E} Z_t^2 - \mathbb{E} u_t^2 = 0$.

i.e. $Z_t \in \overline{sp}\{Y_s,\, s\le t\} \subseteq \overline{sp}\{X_s,\, s\le t\}$.

Rmk: i) $Z_t \in \overline{sp}\{X_s,\, s\le t\}$ is weaker than invertible. Since Z_t may not be written in $\sum_{i\ge 0} \pi_i X_{t-i}$

(e.g. $\exists (X_{kn}) \to \tilde{X} \notin \overline{sp}\{X_s,\, s\le t\}$)

ii) If $\phi \ne 0$ on $|z|<1$, additionally, we have: $\overline{sp}\{X_s,\, s\le t\} = \overline{sp}\{Z_s,\, s\le t\}$. Set $\tilde{P}_{t-1}$ is proj. on it

$\Rightarrow X_t - \tilde{P}_{t-1} X_t = Z_t$ by prop. above

and $\tilde{P}_{t-1} Z_t = 0$. ($Z_t \perp \{Z_s,\, s<t\}$).

Prop: (converse)

If $Z(\theta) \cap Z(\phi) = \emptyset$. $\phi \neq 0$ on $|z|=1$. But $\theta(z)$ has zeros inside $\{|z|<1\}$. Then: $Z_t \notin \overline{sp}\{X_s, s \le t\}$.

Pf: As above. $\exists \tilde{\phi}, \tilde{\theta}$. has no zeros in $|z|<1$.

$u_t \sim WN(0, \sigma^2(\prod_{j=1}^{p}|a_k|)^2(\prod_{j=1}^{q}|b_k|)^2)$

s.t. $\tilde{\phi}(B)X_t = \tilde{\theta}(B)u_t$.

With Rmk ii): $u_t = X_t - \hat{P}_{t-1}X_t$.

Let $\theta(z) = \theta_1(z)\cdot\theta_2(z)$. $Z(\theta_2) \subset \{|z|<1\}$.

$\Rightarrow \tilde{\phi}(B)\theta_1(B)Z_t = \sum_{\mathbb{Z}} \xi_j u_{t-j}$.

(ξ_j) is coefficient of Laurent expansion of $\frac{\phi(z)\tilde{\theta}(z)}{\theta_2(z)}$. $\exists j = -j_0$. s.t. $j_0 > 0$. $\xi_j \neq 0$.

Then: $\langle \tilde{\phi}(B)\theta_1(B)Z_t, u_{t+j_0} \rangle = \xi_{-j_0} \neq 0$.

So: $Z_t \notin \overline{sp}\{X_s, s \le t\}$.

② Approxi. of density:

Thm. If f is sym. posi. spectral density on $[-\pi, \pi]$.

Then. $\forall \varepsilon > 0$. $\exists p \in \mathbb{Z}^+$. $a(z) = \prod_1^p (1 - \eta_i^{-1} z)$. $|\eta_i| > 1$.

$|A|a(e^{-i\lambda})|^2 - f(\lambda)| < \varepsilon$. $\forall \lambda \in [-\pi, \pi]$. $A = \int_{-\pi}^{\pi} f(\lambda)d\lambda / 2\pi(1+a_1^2+\cdots a_p^2)$

where $a(z) = 1 + a_1 z^1 + \cdots a_p z^p$

Rmk: It works for all real valued process (X_t).

Cor. Under the conditions above.

There $\exists$ invertible MA(q)

$X_t = a(B) Z_t$. St.

$|f_X(\lambda) - f(\lambda)| < \varepsilon$, $\forall \lambda$.

Cor. As above. $\exists$ causal AR(p).

$a(B) X_t = Z_t$. St.

$|f_X(\lambda) - f(\lambda)| < \varepsilon$. $\forall \lambda$

(3) Spectral Decomp.:

① Ortho. increment:

Def: Orthogonal increment process is a complex valued sto-process $(Z(\lambda))_{-\pi}^{\pi}$.

St. i) $\mathbb{E}(Z(\lambda)^2) < \infty$. ii) $\mathbb{E}(Z(\lambda)) = 0$.

iii) $\langle Z(\lambda_4) - Z(\lambda_3), Z(\lambda_2) - Z(\lambda_1) \rangle = 0$.

for $\forall (\lambda_1, \lambda_2] \cap (\lambda_3, \lambda_4] = \emptyset$.

Next, we assume $\forall$ o.i.p are right-cont:

i.e. $\lim_{\delta \to 0^+} \| Z(\lambda+\delta) - Z(\lambda) \|^2 = 0$. $\forall \lambda$.

prop. $\forall (Z(\lambda))_{-\pi}^{\pi}$. There $\exists$ unique d.f. $F(\lambda)$.

St. $F(\lambda) = 0$. $\forall \lambda \le -\pi$. $F(\lambda) = F(\pi)$. $\forall \lambda \ge \pi$

$F(\mu) - F(\lambda) = \| Z(\mu) - Z(\lambda) \|^2$. $\mu \ge \lambda$.

Pf: Def: $F(\mu) = \| Z(\mu) - Z(-\pi) \|^2$.

$F(\mu) = F(\lambda) + \| Z(\mu) - Z(\lambda) \|^2$

$\overset{\text{orth.}}{\geq} F(\lambda)$. So $F \uparrow$.

With condition $\Rightarrow$ F is d.f.

Rmk: We have: $\mathbb{E}(dZ(\lambda)\, d\overline{Z(\mu)}) = \delta_{\mu=\lambda}\, dF(\lambda)$.

③ Construction:

i) Consider: $I: D = \{ f(\lambda) = \sum_1^n f_i\, I_{(\lambda_i, \lambda_{i+1}]}(\lambda) \} \subset L^2(F)$

$\longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$. Linear function.

Define by $I(\sum_1^n f_i\, I_{(\lambda_i, \lambda_{i+1}]}) = \sum_1^n f_i\, (Z_{\lambda_{i+1}} - Z_{\lambda_i})$.

It also satisfies: $\langle I(f), I(g) \rangle_{L^2(\Omega)} = \langle f, g \rangle_{L^2(F)}$.

So: I is an isometry. extend I to $\overline{D} = L^2(F)$.

Then: $\forall f \in L^2(F)$. $\int_{-\pi}^{\pi} f\, dZ(\lambda) \overset{\text{Def}}{:=} I(f)$.

ii) Conversely, consider $T: \mathcal{H} = \overline{LS}\{X_t\}_{\mathbb{Z}} \subset L^2(\Omega)$

$\overset{LF}{\longrightarrow} \mathcal{K} = \overline{LS}\{e^{it\lambda}\}_{t \in \mathbb{Z}} \subset L^2(F)$.

Define by $T(\sum_1^n a_i X_{t_i}) = \sum_1^n a_i e^{it_i\lambda}$.

where (X_t) is fixed zero-mean stationary r.v.'s.

Prop. T is well-def.

$\underline{Pf}$: $\| T(\sum_1^n a_i X_{t_i}) - T(\sum_1^m b_k X_{\tau_k}) \|^2_{L^2(F)}$

$$= \| \sum_1^n a_i e^{it_i\lambda} - \sum_1^m b_k e^{i\tau_k\lambda} \|^2_{L^2(F)}$$

$$\overset{\text{by i)}}{=} \mathbb{E} | \sum_1^n a_i X_{t_i} - \sum_1^m b_k X_{\tau_k} |^2$$

Note T is also an isomorphism. We extend

T to: $\bar{H} = \overline{LS}\{X_s\} \to \bar{K} = L^2(F)$.

So, we have:

$\underline{\text{Thm.}}$ For F is spectral d.f of stationary

seq. (X_t). Then: $\exists$ unique isomorphism.

$T : \overline{LS}\{X_t\}_{\mathbb{Z}} \to L^2(F)$. St. $T(X_t) = e^{it\lambda}$.

$\underline{\text{Cor.}}$ $Z(\lambda) = T^{-1}(I_{(-\pi,\lambda]}(\mu))$. is o.i.p.

St. its associated d.f is F.

$\underline{Pf}$: Check by isometry: $\langle \cdot, \cdot \rangle_{L^2(F)} = \langle \cdot, \cdot \rangle_{L^2(\Omega)}$

③ $\underline{\text{Thm.}}$ (representation)

For F is spectral d.f of stationary r.v.'s

$(X_t)_{\mathbb{Z}}$ Then: $\exists$ unique o.i.p. $Z(\lambda)$. St.

i) right-conti. ii) $X_t = \int_{-\pi}^{\pi} e^{it\lambda}\, dZ(\lambda)$ a.s.

iii) $\mathbb{E}|Z(\lambda) - Z(\mu)|^2 = F(\lambda) - F(\mu)$. $\forall \lambda \geq \mu$.

Pf: From (X_t). $\Rightarrow$ we have isomorphism T

$\Rightarrow$ we have $(Z(\lambda))_{\lambda\in[-\pi,\pi]}$ $\Rightarrow$ iso. I.

So: $X_t = I(e^{it\lambda}) = \int_{-\pi}^{\pi} e^{it\lambda} \, dZ(\lambda)$.

since we can check: $IT = 1d$.

Cor. $Y_t \in \overline{LS}\{X_s, s\in\mathbb{Z}\} \iff \exists f \in L^2(F)$.

$Y_t = \int_{-\pi}^{\pi} e^{it\lambda} \, dZ(\lambda)$. F is spectral d.f. of (X_t).

Prop. If spectral d.f. F has discontinuities (λ_k).

Then $X_t = \int_{(-\pi,\pi]/\{\lambda_k\}} e^{it\nu} \, dZ(\nu) + \sum_k (Z(\lambda_k) - Z(\lambda_k-)) e^{it\lambda_k}$

and $Var(Z(\lambda_k) - Z(\lambda_k-)) = F(\lambda_k) - F(\lambda_k-)$.

④ Inversion Formula:

Thm. For (X_n) zero-mean stationary seq with $\gamma(\cdot)$.

We have: $\frac{1}{2\pi}\sum_{k\le|n|} X_k \alpha_k \xrightarrow[L^2]{n\to\infty} Z(w) - Z(v)$.

$\frac{1}{2\pi}\sum_{k\le|n|} \gamma(k)\alpha_k \xrightarrow[L^2]{n\to\infty} F(w) - F(v)$

where $\alpha_k = \int_{v}^{w} e^{-ik\lambda} \, d\lambda$.

Pf: $\sum_{k\le|n|} \alpha_k e^{ik\lambda} \xrightarrow{L^2} I_{(v,w]}(\lambda)$. Apply T^{-1} on the

Thm. (Continuous case)

$\gamma(s) = \int_{-\infty}^{+\infty} e^{is\lambda} dF(\lambda)$. $X_t = \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda)$.

And for continuity v, w of F, we have:

$\frac{1}{2\pi} \int_{-T}^{T} \left(\int_v^w e^{-it\eta} d\eta \right) \gamma(t) \xrightarrow[T\to\infty]{L^2} F(w) - F(v)$.

$\frac{1}{2\pi} \int_{-T}^{T} \left(\int_v^w e^{-it\eta} d\eta \right) X(t) \xrightarrow[T\to\infty]{L^2} Z(w) - Z(v)$.

(4) Linear Filters:

Def: $C = (c_{t,k})_{t,k\in\mathbb{Z}}$ is a linear filter. It's time-invariant if $c_{t,k} = h_{t-k}$; it's causal if $h_k = 0 \ \forall k < 0$.

Thm. (X_t) is zero-mean stationary seq with repr.: $X_t = \int_{-\pi}^{\pi} e^{itv} dZ_X(v)$. If $H = (h_i)$ is time-invariant linear filter. s.t. $\sum_{-n}^{n} e^{-ij\lambda} h_j \xrightarrow{L^2(F)} h(e^{-i\lambda})$. Then we have: $Y_t = \sum_{\mathbb{Z}} h_i X_{t-i}$ is stationary st. with $F_Y(\lambda) = \int_{-\pi}^{\lambda} |h(e^{-iv})|^2 dF_X(v)$. and $Y_t = \int_{-\pi}^{\pi} e^{itv} h(e^{-iv}) dZ_X(v)$.

Besides, $X_t = \int_{-\pi}^{\pi} e^{itv} / h(e^{-iv}) \, dZ_Y(v)$. if $h \neq 0$ on $|z| = 1$.

Rmk: $|h(e^{-i\lambda})|$ is amplified gain of H.